PRACTICAL TRIGONOMETRY

PLAYNE
AND
FAWDRY

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PREFACE.

DURING the last few years a great change has come over the teaching of Elementary Mathematics. The laborious months hitherto spent in acquiring skill in the manipulation of elaborate Algebraical and Trigonometrical transformations have often given the beginner a dislike for Mathematics and have retarded his progress.

It has been shown that it is quite possible to arrange (for the average student) a course of Mathematics which is both interesting and educational, by constantly keeping before him the practical application of the subject, and omitting as much as possible those parts of Mathematics which are purely academical. The object of this book is to give the reader such a working knowledge of elementary Trigonometry, without avoiding the difficulties or sacrificing thoroughness. Much that has hitherto been found in the text-books has been omitted, and the examples throughout will be seen to be more practical than is usually the case.

The book contains many and varied examples to be worked out by the student, but we have avoided the grouping together of batches of examples of the same type, believing that such a system is the cause of much mechanical and unintelligent work. Collections of miscellaneous examples occur frequently, so that the student may be constantly revising what he has learnt in the earlier chapters. We have avoided those artificial questions which have gradually been evolved by the

ingenuity of examiners, but are never met with in the practical application of Mathematics, and have introduced as many examples as possible to illustrate the use of Trigonometry in Mechanics, Physics and Analytical Geometry. In numerical work we have indicated the degree of accuracy to which the results are reliable.

Enough examples are worked out in the text to show how each new principle may be applied, and to show the best way of arranging the work—which is of especial importance when logarithms are used; but we have endeavoured to leave the student as much as possible to his own intelligence.

Another special feature of the book is Chapter X, which deals with solid figures and angles which are not in one plane. We have also added an introduction to Trigonometrical Surveying.

We believe that the book will be of value to those who are preparing for Army and Civil Service Examinations, to Technical Students, and to all who require Trigonometry for practical purposes.

Our best thanks are due to several friends and colleagues for much kind help, and in particular to Mr G. W. Palmer of Clifton College.

To meet the wishes of many teachers who use this book we have added an appendix containing a considerable number of Identities.

H. C. P.
R. C. F.

August 1914.

CONTENTS.

CHAPTER I.

ANGLES.

1. LET OX be a fixed straight line, and let a straight line OP, initially coincident with OX, turn about the point O in one plane; then, as it turns, it is said to describe the angle XOP. The magnitude of the angle depends on the amount of revolution which OP has undergone.

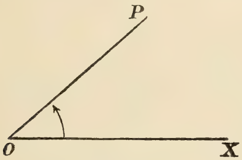

Fig. 1.

OX is called the *initial line.*

In Trigonometry there is no limit to the magnitude of the angles considered.

When OP reaches the position OX′, i.e. when X′OX is a straight line, it has turned through an angle equal to two

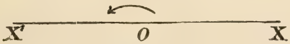

Fig. 2.

right angles; and when it again becomes coincident with OX it has turned through four right angles.

2. Sexagesimal Measure.

Since all right angles are equal, a right angle might be chosen as the unit of measurement of angles but it is too large to be convenient. The unit selected is one-ninetieth part of a right angle and is called a *degree* (1°).

A degree is subdivided into 60 equal parts, each of which is called a *minute* ($1'$), and a minute into 60 equal parts, each of which is called a *second* ($1''$).

Thus $15^\circ\,42'\,27''$ is read 15 degrees, 42 minutes, 27 seconds.

This system of measurement of angles is called the *Sexagesimal measure.*

Another unit, called a *Radian*, is used especially in theoretical work and will be discussed in Chap. IX.

Example (i).

The angle subtended at the centre of a circle by the side of an inscribed regular figure may readily be expressed in Sexagesimal Measure.

Let the regular figure be a Pentagon.

Then at the centre O we have five equal angles whose sum is four right angles;

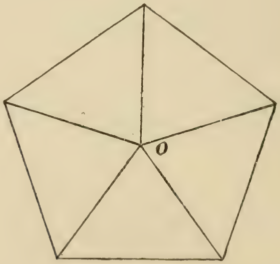

Fig. 3.

$\therefore$ the angle subtended by each side $= \dfrac{360^\circ}{5} = 72^\circ$.

Example (ii).

The angle of a regular figure, e.g. an octagon, may be found thus :—

Join any angular point A to the other angular points.

Six triangles are formed, the sum of all their angles being 12 right angles.

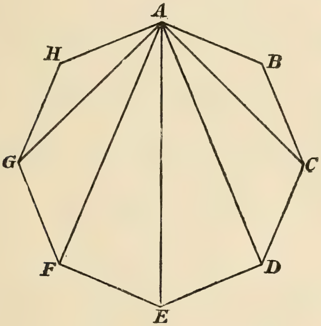

Fig. 4.

But these angles make up the eight angles of the figure ;

$\therefore$ each angle of the figure $= \frac{1080°}{8} = 135°$.

Or we may make use of the geometrical theorem that all the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

Thus if a figure has n sides, the interior angles make up $2n-4$ right angles.

If the figure is regular, each angle is $\frac{2n-4}{n}$ right angles.

Examples. I.

1. Express in degrees the angles of an equilateral triangle.

2. One angle of a right-angled triangle is $37° 15' 20''$, find the other acute angle.

3. Two angles of a triangle are $42° 14'$ and $100° 12'$, find the other angle.

4. What are the angles between the two hands of a clock at 5 o'clock?

5. Express in degrees the angles between the two hands of a clock at 6.15.

6. Through how many degrees does the minute hand of a clock turn between 3.10 and 7.25?

7. Express $27° 14' 5''$ in seconds.

8. Find the sexagesimal measure of ·486 of a right angle.

9. Find to the nearest second the angle of (i) a regular hexagon, (ii) a regular heptagon, (iii) a regular pentagon.

10. Express $51° 17' 45''$ as a decimal of a right angle to 5 places of decimals.

CHAPTER II.

TRIGONOMETRICAL FUNCTIONS.

3. Note on Similar Triangles.

Two equiangular triangles are proved in Geometry to have their corresponding sides proportional, and the triangles are called Similar.

That is to say if ABC, A′B′C′ are two triangles in which the angles at A, B, C respectively equal those at A′, B′, C′, then

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

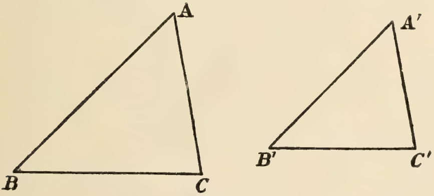

Fig. 5.

Conversely, if

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'},$$

the two triangles ABC, A′B′C′ are equiangular, having those angles equal which are opposite corresponding sides.

The student who is unfamiliar with the properties of similar triangles should carefully work through the following Exercise.

Draw an angle XOR equal to 50°. Take any three points P, P_1, P_2, on OR. From these points drop perpendiculars PN, P_1N_1, P_2N_2, on OX. Measure these perpendiculars and

the lengths ON, ON_1, ON_2. Then write down the values of the following ratios correct to 2 decimal places:

$$\frac{NP}{OP}, \frac{N_1P_1}{OP_1}, \frac{N_2P_2}{OP_2}; \frac{ON}{OP}, \frac{ON_1}{OP_1}, \frac{ON_2}{OP_2}, \frac{NP}{ON}, \frac{N_1P_1}{ON_1}, \frac{N_2P_2}{ON_2}.$$

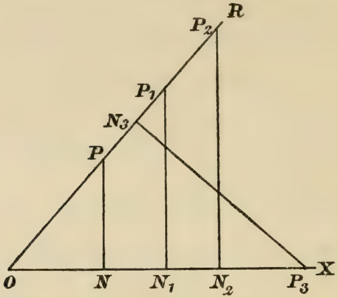

Fig. 6.

Now take a point P_3 on OX and drop a perpendicular P_3N_3 on OR. Measure OP_3, P_3N_3, ON_3 and find the values of

$$\frac{N_3P_3}{OP_3}, \frac{ON_3}{OP_3}, \frac{N_3P_3}{ON_3}.$$

State what conclusions you draw from your results.

4. Trigonometrical Functions.

Let XOR be any angle θ. From any point P in one of the boundary lines of the angle draw PN perpendicular

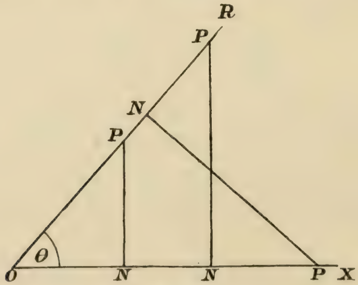

Fig. 7.

to the other boundary line. From the properties of similar

triangles, or by actual measurement, it may be shown that the ratios

$$\frac{NP}{OP}, \quad \frac{ON}{OP}, \quad \frac{NP}{ON}$$

are constant for all positions of P so long as the magnitude of the angle remains unchanged.

These ratios, which depend only on the magnitude of θ are called respectively the *sine*, *cosine*, *tangent* of θ, and their reciprocals are called respectively the *cosecant*, *secant*, *cotangent* of θ.

They are thus abbreviated :

$$\sin\theta = \frac{NP}{OP}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{OP}{NP};$$

$$\cos\theta = \frac{ON}{OP}, \quad \sec\theta = \frac{1}{\cos\theta} = \frac{OP}{ON};$$

$$\tan\theta = \frac{NP}{ON}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{ON}{NP}.$$

Note. In view of a distinction in *Sign* which will be made in Chap. IV. between the direction NP and the direction PN, it is preferable here to write NP and not PN in the expressions for $\sin\theta$ and $\tan\theta$.

Calling NP the side opposite the angle θ, ON the side adjacent to θ, and OP the hypotenuse, we may write them

$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}};$$

$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}};$$

$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta};$$

and similarly for $\operatorname{cosec}\theta$, $\sec\theta$ and $\cot\theta$.

These ratios are called the trigonometrical functions or ratios of the angle θ.

Note. $(\sin A)^2$ is written $\sin^2 A$; i.e. if

$$\sin A = \tfrac{2}{3}, \quad \sin^2 A = \tfrac{4}{9}.$$

5. The definitions of the trigonometrical functions still hold good for an angle greater than 90°.

If from a point P in one of the boundary lines of the

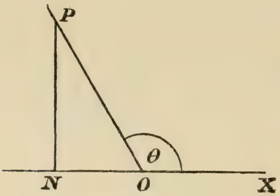

Fig. 8.

angle θ, PN be drawn perpendicular to the other boundary line produced if necessary, then

$$\sin\theta = \frac{NP}{OP};$$

$$\cos\theta = \frac{ON}{OP};$$

$$\tan\theta = \frac{NP}{ON} \text{ etc.}$$

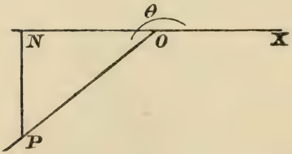

Fig. 9.

For the present we shall confine our attention to acute angles, and it will be explained in Chap. IV. that there are certain conventions of sign to be adopted in treating of the ratios of angles greater than a right angle.

6. Variation in the value of the ratios as the angle increases.

In order to compare the values of fractions in Arithmetic it is convenient to express them with the same denominator, so in Trigonometry we can compare the values of various ratios by keeping OP (called the radius vector) of constant length.

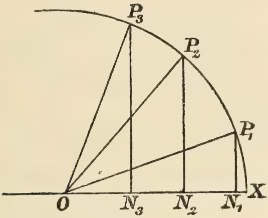

Fig. 10.

As the angle XOP increases from 0° to 90°, does sin A increase in value or diminish?

Discuss what happens to the other trigonometrical ratios.

Why is sin A not greater than 1? What is the greatest value of cos A? Can tan A exceed 1?

Note. The angles of a triangle ABC are conveniently denoted A, B, C, and the sides opposite these angles respectively a, b, c.

Examples. II a.

1. ABC is a triangle, B being a right angle, AC $=5''$, AB $=4''$. Calculate the length of BC and write down sin A, cos C, tan A, sec A, cosec C. Find the value of $\sin^2 A+\cos^2 A$, $\dfrac{\sin A}{\cos A}$, tan A, $1+\tan^2 C$, $\sec^2 C$.

2. In a triangle, $c=17$, $a=8$, $b=15$; prove that $B=90°-A$. Write down the values of sin A, sin B, tan B, cos A, cot A, cosec B, $\cos(90°-A)$, $\sin(90°-A)$, $\tan(90°-A)$. What ratio of A is equal to (i) $\cos(90°-A)$, (ii) $\sin(90°-A)$, (iii) $\tan(90°-A)$, (iv) $\dfrac{\sin A}{\cos A}$?

3. The hypotenuse c of a right-angled triangle is 15″ and the side $a=12''$. Calculate the length of b. Find the values of $1+\tan^2 A$, $\sec^2 A$, $1+\tan^2 B$, $\sec^2 B$, $1+\cot^2 A$, $\operatorname{cosec}^2 A$, $\sin^2 A+\cos^2 A$, $\sin^2 B+\cos^2 B$.

4. In the triangle ABC, $A=90°$, and AD is drawn perpendicular to BC. From the triangles ABD and ABC write down two values of $\sin B$, and two values of $\cos B$. Hence find AD and BD if $a=41$, $c=40$, $b=9$.

5. A point A on the circumference of a circle is joined to BC the extremities of the diameter. AD is drawn perpendicular to BC. Prove that $\angle BAD = C$, and $\angle DAC = B$.

From the triangles ABC and ABD write down two values of $\sin C$. Hence prove $AB^2 = BC \cdot BD$.

Prove in a similar way that $AC^2 = BC \cdot CD$.

6. In the same figure from the triangles ABD, ADC write down two values for $\cot B$. Hence prove $AD^2 = BD \cdot DC$.

7. ABC is any triangle, AD, BE, CF are the perpendiculars drawn from the angular points to the opposite sides.

Prove (i) $\dfrac{AD}{AB}=\dfrac{FC}{BC}$, (ii) $\dfrac{FC}{AF}=\dfrac{BE}{AE}$, (iii) $\dfrac{CD}{AC}=\dfrac{EC}{BC}$.

8. AB is the diameter of a circle, C a point on the circumference. The tangent at B meets AC produced at D.

Prove $$\angle CBD = \angle CAB.$$

From the triangles ACB and BCD write down two values of $\tan A$. Hence prove $BC^2 = CA \cdot CD$.

9. In the same figure prove: (i) $\dfrac{DB}{BA}=\dfrac{BC}{CA}$,

(ii) $\dfrac{AB}{AD}=\dfrac{BC}{BD}=\dfrac{AC}{AB}$, (iii) $DB^2 = AD \cdot DC$.

10. ABCD is a rectangle, $AD=12''$, $AB=5''$. Draw AE perpendicular to BD. Write down two values for $\sin ADE$. Hence find the length of AE.

7. Geometrical constructions for trigonometrical ratios with given angles.

It will be found useful to employ squared paper for these examples, and generally to write the ratio in the form of a fraction with 10 as its denominator.

Example (i).

Draw an angle of 49° and find from measurements the value of sin 49°.

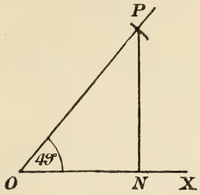

Fig. 11.

Draw the angle XOP by means of a protractor: since the hypotenuse is to be the denominator, mark off OP = 10 units and draw PN perpendicular to OX.

Then $$\sin 49° = \frac{NP}{OP} = \frac{7·5}{10} = ·75.$$

Example (ii).

Draw an angle of 54°, and find from the drawing sec 54°.

Draw the angle XOP = 54°.

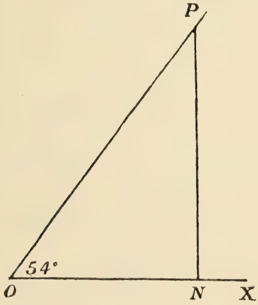

Fig. 12.

Mark off ON = 10 units.

Erect the perpendicular NP.

Then $$\sec 54° = \frac{OP}{ON} = \frac{17}{10} = 1·7.$$

8. Geometrical construction for angles with given ratios.

Inverse Notation. The angle whose sine is x is written $\sin^{-1}x$.

Thus $\cos^{-1}\cdot 86$ is read "the angle whose cosine is ·86."

Example.

To construct an angle whose sine is ·72, that is $\sin^{-1}\cdot 72$.

Since $\cdot 72 = \frac{7\cdot 2}{10}$, draw two lines PN and ON at right angles. Mark off PN $= 7\cdot 2$ units and with centre P, radius 10 units, strike an arc PO.

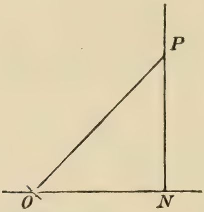

Fig. 13.

Then $\angle$ PON is $\sin^{-1}\cdot 72 = 46°$.

9. The trigonometrical ratios of 60°, 30°, 45°.

These ratios may be found by Geometrical reasoning without accurate drawing.

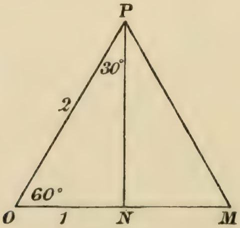

Fig. 14.

(i) If $\angle \mathrm{PON} = 60^\circ$, then $\angle \mathrm{OPN} = 30^\circ$.

Complete the equilateral triangle OPM.

Then if $\quad \mathrm{OP} = 2$ units,

$\mathrm{ON} = 1$ unit;

and $\quad \mathrm{OP}^2 = \mathrm{PN}^2 + \mathrm{NO}^2$;

$\therefore\ \mathrm{PN} = \sqrt{3}$ or $1{\cdot}732$;

$$\therefore\ \sin 60^\circ = \frac{\sqrt{3}}{2} \text{ or } {\cdot}866;\ \cos 60^\circ = \frac{1}{2} \text{ or } {\cdot}5;$$

$$\tan 60^\circ = \sqrt{3} \text{ or } 1{\cdot}732.$$

(ii) From the same triangle, since $\angle \mathrm{OPN} = 30^\circ$,

$$\sin 30^\circ = \frac{1}{2} \text{ or } {\cdot}5;\ \cos 30^\circ = \frac{\sqrt{3}}{2} \text{ or } {\cdot}866,$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1{\cdot}732}{3} = {\cdot}577.$$

(iii) If $\angle \mathrm{PON} = 45^\circ$, then $\angle \mathrm{OPN} = 45^\circ$;

$\therefore$ if $\mathrm{PN} = \mathrm{ON} = 1$ unit,

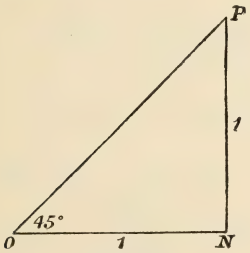

Fig. 15.

since $\quad \mathrm{OP}^2 = 1 + 1 = 2;\ \therefore\ \mathrm{OP} = \sqrt{2}$.

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1{\cdot}414}{2} = {\cdot}707,$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = {\cdot}707,$$

$$\tan 45^\circ = 1.$$

10. Relations between the Trigonometrical Ratios.

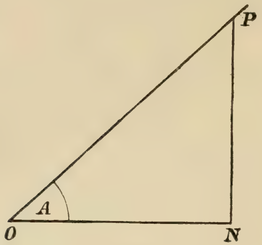

Fig. 16.

(1) To prove $\dfrac{\sin A}{\cos A} = \tan A$.

$$\frac{\sin A}{\cos A} = \frac{\dfrac{NP}{OP}}{\dfrac{ON}{OP}} = \frac{NP}{ON} = \tan A.$$

Similarly $\dfrac{\cos A}{\sin A} = \dfrac{1}{\tan A} = \cot A$.

(2) To prove $\sin^2 A + \cos^2 A = 1$.

$$\sin^2 A + \cos^2 A = \frac{NP^2}{OP^2} + \frac{ON^2}{OP^2} = \frac{NP^2 + ON^2}{OP^2}$$

$$= \frac{OP^2}{OP^2} = 1.$$

(3) Prove in a similar manner

$$\sec^2 A = 1 + \tan^2 A,$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A.$$

Relations such as these which are true for all angles are called *Identities.*

11. Given one ratio of an angle to find the other ratios.

If it is not required to find the angle, the ratios may be calculated without accurate drawing.

Example.

Given $\sin A = \frac{8}{17}$, to find the other trigonometrical ratios of A.

If $PN = 8$ units and $OP = 17$ units, then

$$17^2 = 3^2 + ON^2,$$

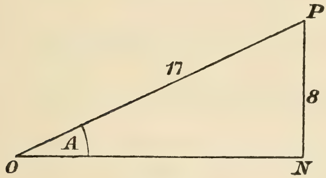

Fig. 17.

$$\therefore\ ON^2 = 17^2 - 8^2 = (17+8)(17-8)$$
$$= (25)(9),$$
$$\therefore\ ON = 15,$$
$$\therefore\ \cos A = \tfrac{15}{17},\quad \tan A = \tfrac{8}{15},\ \text{etc.}$$

Or using the result of Article 10 (2), we have

$$\cos A = \pm\sqrt{1 - \sin^2 A} = \pm\sqrt{1 - \frac{8^2}{17^2}} = \pm\frac{15}{17}.$$

We shall disregard the negative sign until Chap. IV.

Examples. II b.

1. Draw an angle of 37°. Find its ratios by measurement to two decimal places.

2. Draw an angle of 49°. Find by measurement $\sin 49°$, $\cos 49°$, $\sec 49°$, $\tan 49°$; with your results test the following, $\sin^2 49 + \cos^2 49 = 1$, $\sec^2 49 - \tan^2 49 = 1$.

3. Construct the angle whose cosine is ·52; measure it, and find its sine and tangent.

4. Given that $\sin A = \frac{6}{7}$, calculate the value of $\sec A$ and $\tan A$ to two decimal places. Using your results, find by how much $\sec^2 A$ differs from $1 + \tan^2 A$.

5. Given that $\operatorname{cosec} A = \frac{5}{4}$, find the values of $(\sin A + \cos A)^2$ and $\sin^2 A + \cos^2 A$.

6. Draw the angle A whose tangent is 6. Bisect this angle and find by measurement $\tan \frac{A}{2}$. By how much does it differ from $\frac{1}{2} \tan A$?

7. Construct the angle whose cosecant is 2·14. Measure it and find its cosine and secant to one decimal place.

8. Draw an angle of 40°. Find its tangent. Bisect the angle and from measurements find tan 20°. From the same diagram find cot 70°.

9. If $\sin \theta = \cdot 5$, find the value of $1 + \tan^2 \theta$.

10. If $\sin \theta = \frac{p}{q}$, prove that $\cos \theta = \frac{\sqrt{q^2 - p^2}}{q}$.

11. The diagonal of a rectangle is twice one of the sides: prove that the ratio of the sides is $\sqrt{3} : 1$.

12. ABC is a right-angled triangle with BA = BC. BD is drawn perpendicular to AC. Prove that $\frac{BD}{BC} = \frac{1}{\sqrt{2}}$ and that BD = DC.

13. ABCDEF is a regular hexagon. If AB = 4″, find the lengths of BE and BF.

Miscellaneous Examples. A.

1. Draw with your protractor an angle of 142°, also one of 210°.

2. Draw an angle of 48°. From measurements of your drawing find tan 48°.

3. Draw a triangle ABC having B a right angle, $b = 15$, $c = 12$. Write down sin A, cos C. What relation is there between the angles A and C?

4. Find the number of degrees in the angle of a regular hexagon. Prove that the side of a regular hexagon equals the radius of the circumscribing circle.

5. Express ·2145 of a right angle in degrees, minutes and seconds.

6. Construct the angle whose tangent is 2, and prove that its sine is $\frac{2\sqrt{5}}{5}$.

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7. The angle subtended by a side of a regular figure at the centre of its inscribed circle is 36°. How many sides has the figure?

8. Draw carefully the angle whose cosine is ·37. From measurements find the cotangent of the angle.

9. What decimal of a right angle is $52° 12'$?

10. An isosceles triangle has each of its equal sides double the base; find the cosine and cotangent of the base angles.

11. Find the angle of a regular figure of 12 sides.

12. If $x = a \cos \phi$ and $y = b \sin \phi$, prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

---

13. Draw accurately the angle $\operatorname{cosec}^{-1} 2{\cdot}4$; also the angle $\operatorname{cosec}^{-1} 1{\cdot}2$. Measure the angles and find their difference in degrees.

14. A diameter AB of a circle bisects the chord CD at O. If $\sin \text{ABC} = \frac{4}{5}$ and $\text{AC} = 10''$, find AO.

15. Given $\sec \text{A} = \frac{13}{5}$, calculate $\tan \text{A}$. Show that for this angle $\sin^2 \text{A} = 1 - \cos^2 \text{A}$.

16. Two tangents OA, OB are drawn to a circle of radius 5″ from a point 12″ from the centre C. Prove that $\sin \text{CAB} = \frac{5}{12}$ and hence that the distance of C from $\text{AB} = 2\frac{1}{12}''$.

17. The three angles of a right-angled triangle are such that $2\text{B} = \text{A} + \text{C}$; find them in degrees.

18. Prove that $\sin 60° = 2 \sin 30° \cos 30°$,

and that $\cos 60° = \cos^2 30° - \sin^2 30°$.

CHAPTER III.

THE USE OF FOUR FIGURE TABLES.

12. The values of the Trigonometrical Ratios will be found in Bottomley's 4-figure Tables, pp. 32—43.

The ratios are given at intervals of 6 minutes with difference columns for variations of 1, 2, 3, 4, 5 minutes.

Since all the sines and cosines are $\not> 1$ the values of these ratios are entirely decimal, and the decimal points are not printed; but in all other ratios the decimal point and any integral part is printed in the first column only.

Note that as the angle increases from 0° to 90° the *co*sine, *co*tangent, and *co*secant diminish (see Chap. II. § 6).

Example (i).

To find the value of sin 31° 47′.

The following is an extract from the table of Natural Sines on p. 32 of Bottomley's tables.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1 2 3	4 5
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2 5 7	10 12

In the row opposite 31° and in the column under 42′ we find 5255.

The difference for 5′ is given in the same row in the last column under 5: we find 12.

Thus $\sin 31° 42' = \cdot 5255$,

difference for $5' = \cdot 0012$;

$\therefore \sin 31° 47' = \cdot 5267$.

The difference is added since the sine increases if the angle increases.

Example (ii).

To find the value of cos 49° 21′.

From the tables

$$\cos 49^\circ 18' = \cdot 6521,$$

difference for $3' = \cdot 0007$;

$$\therefore \cos 49^\circ 21' = \cdot 6514.$$

The difference is subtracted since the cosine diminishes as the angle increases.

Note. The correct value will not be found by taking $\cos 49^\circ 24'$ from the tables and using the difference table.

Example (iii).

To find the angle whose cotangent is 4·8142.

Since the difference column is to be subtracted we find the nearest angle with a cotangent greater than 4·8142.

The bar over the figures in the tables denotes that the whole number has changed in the row and in this case is no longer 5 but 4.

Thus $\cot 11^\circ 42' = 4 \cdot 8288,$

difference for $2' = \ \cdot 0148$;

$$\therefore \cot 11^\circ 44' = 4 \cdot 8140,$$

i.e. the angle $\cot^{-1} 4 \cdot 8142$ is $11^\circ 44'$ to the degree of accuracy given by the tables.

Example (iv).

By using the tables we can find angles to satisfy given equations. The identities in Chap. II. § 10 will be found useful in throwing the equation into a form suitable for solving.

Find the acute angles which satisfy the equation

$$3 \operatorname{cosec}^2\theta - 11 \cot\theta + 7 = 0.$$

By using the identity $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ the equation can be written in terms of one unknown,

$$3(1 + \cot^2\theta) - 11 \cot\theta + 7 = 0,$$

$$3 \cot^2\theta - 11 \cot\theta + 10 = 0,$$

$$(3 \cot\theta - 5)(\cot\theta - 2) = 0;$$

$$\therefore \cot\theta = 1 \cdot 6667 \text{ or } \cot\theta = 2;$$

$$\therefore \theta = 30^\circ 58' \text{ or } \theta = 26^\circ 34' \text{ from the tables.}$$

Examples. III a.

Look up in the tables,

1. $\sin 19°$.	2. $\sin 33° 22'$.	3. $\cos 65° 4'$.
4. $\cos 18° 5'$.	5. $\cot 30° 21'$.	6. $\sin 63° 50'$.
7. $\tan 16° 50'$.	8. $\operatorname{cosec} 14° 31'$.	9. $\sec 70° 10'$.
10. $\operatorname{cosec} 9° 42'$.	11. $\cot 11° 37'$.	12. $\tan 80° 48'$.
13. $\sin^{-1} \cdot 8867$.	14. $\tan^{-1} 2{\cdot}0248$.	15. $\cos^{-1} \cdot 4830$.
16. $\tan^{-1} 5{\cdot}0577$.	17. $\cot^{-1} \cdot 2600$.	18. $\sec^{-1} 4{\cdot}0855$.

Find the acute angles which satisfy the following equations:—

19. $10 \sin^2\theta - 7 \sin\theta + 1 = 0$.   20. $15 \cos\theta + 8 \sec\theta = 22$.

21. $9 \cos^2\theta + 18 \sin\theta = 17$.   22. $4 \sec^2\theta - 17 \tan\theta + 11 = 0$.

13. Right-angled triangles.

It is very important to be able to write down at once the sides of a right-angled triangle in terms of a side and the ratios of a given angle.

Example (i).

Given the side $OP = x$ and the angle $PON = \theta$, PNO being a right angle.

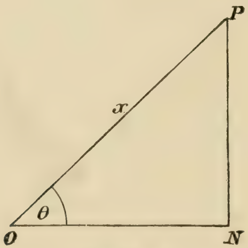

Fig 18.

We have

$$NP = x \times \frac{NP}{x}$$
$$= x \sin\theta.$$
$$ON = x \times \frac{ON}{x}$$
$$= x \cos\theta.$$

Example (ii).

Given BC = 10″ and ∠ABC = 40°, ∠ACB = 90°,

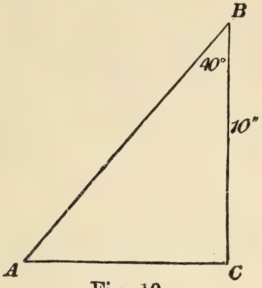

Fig. 19.

$$\mathrm{AC} = 10 \times \frac{\mathrm{AC}}{10}$$
$$= 10 \tan 40°$$
$$= 10 \times \cdot 8391 = 8 \cdot 391.$$
$$\mathrm{AB} = 10 \times \frac{\mathrm{AB}}{10}$$
$$= 10 \sec 40°$$
$$= 10 \times 1 \cdot 3054$$
$$= 13 \cdot 054.$$

Exercise.

Practice writing down the other sides of the following right-angled triangles in terms of the ratios of the given angle and the given side.

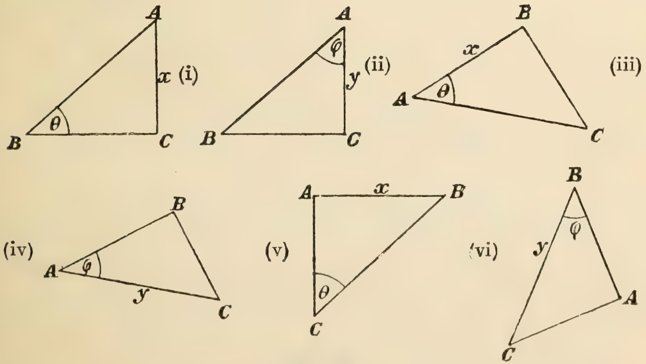

Fig. 20.

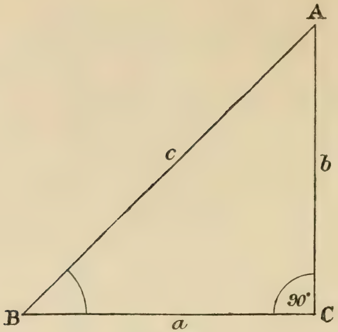

Fig. 21.

Look up the ratios in the tables and write down the lengths of the other sides of the triangle ABC (Fig. 21) from the following data :—

(vii) B $=36°$, $b=10$. (viii) A $=24°$, $b=10$.

(ix) B $=28° 11'$, $a=20$. (x) A $=41° 19'$, $a=10$.

(xi) B $=38° 14'$, $c=25$. (xii) A $=33° 17'$, $c=20{\cdot}25$.

14. Angles of Elevation and Depression.

The angle which a line joining the eye of an observer and a distant object makes with the horizontal plane is

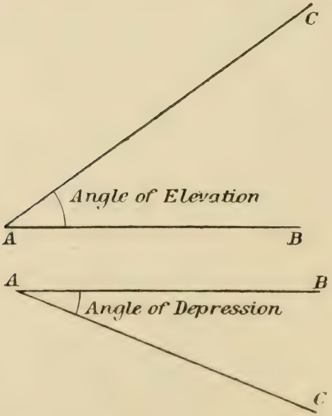

Fig. 22.

called the *Angle of Elevation* if the object be above the observer, and the *Angle of Depression* if the object be below the observer.

Thus in fig. 1 if AB be the horizontal line through A, to the observer at A the angle BAC is the angle of elevation of the point C.

In fig. 2 the angle BAC is the angle of depression of the point C.

Example.

Find the angle of elevation of the sun if the shadow cast by a stick 6 ft. high is 4 ft. 4 in.

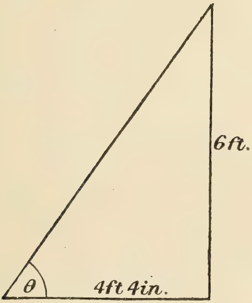

Fig. 23.

Let θ be the angle required; then

$$\tan\theta = \tfrac{72}{52} = 1{\cdot}4;$$

$\therefore$ from the tables $\theta = 54° \, 30'$.

Examples. III b.

1. Find the altitude of an equilateral triangle whose sides are 4″.

2. In the triangle ABC, $A = 90°$, $C = 50°$, $b = 10''$. Draw AD perpendicular to BC and find the lengths of AD, CD, AB, BD.

3. I observe the angle of elevation of the top of a tower 240 feet high to be 37°. What is my horizontal distance from the foot of the tower?

4. Find the angle of elevation of the sun if a tower 212 feet high casts a shadow 327 feet long.

5. The steps of a staircase are 10″ wide and 7″ high. How many degrees are there in the slope of the staircase?

6. AD is the perpendicular from A on the side BC of a triangle ABC. If $B = 32°$, $BD = 7$ ft., $DC = 5$ ft., find AD, AB and the angle C.

7. The angle of depression of a boat from the top of a cliff 200 ft. high is 36° 13′. Find the distance of the boat from the foot of the cliff.

8. The sides of a parallelogram are 4 ft. and 5 ft. and the acute angle between them is 50°. Find the lengths of the perpendicular distances between the parallel sides.

9. Find the lengths of the three perpendiculars from the angular points to the opposite sides of an isosceles triangle whose equal sides are 12 cms. and the included angle 40°.

10. From the top of a spire the angle of depression of an object 100 feet from its base is 56°; find the height of the spire.

11. In a triangle ABC, $B = 70°$, $C = 50°$, $c = 20''$. Draw AE perpendicular to BC and BD perpendicular to AC. Find the lengths of BD, BE, AE, AC.

12. From a point 500 feet from its base the angle of elevation of a tower is 26° 11′. Find the height of the tower.

13. ABCD is a quadrilateral inscribed in a circle of 10 ft. radius. If AC is a diameter and $\angle ABD = 15°$, $\angle ACB = 40°$, find the lengths of the sides of the quadrilateral.

15. Illustrative Examples.

In the following examples the angles are assumed to be acute, but it will be shown in Chap. v. that the theorems are true also when the angles are obtuse.

Example (i).

Prove that the area of a triangle $= \frac{1}{2}$ product of two sides × sine of included angle.

We have, area of triangle $(\Delta) = \frac{1}{2}ap$, when p is the perpendicular on the side a from the opposite angular point.

But $$p = b \times \frac{p}{b}$$
$$= b \sin \mathrm{C};$$
$$\therefore \quad \Delta = \tfrac{1}{2} ab \sin \mathrm{C}.$$

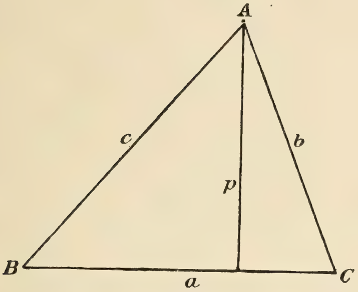

Fig. 24.

Exercise.

(1) Prove also that

$$\Delta = \tfrac{1}{2} ac \sin \mathrm{B} = \tfrac{1}{2} bc \sin \mathrm{A}.$$

(2) Find a formula for the area of a parallelogram in terms of two adjacent sides and the included angle.

(3) Show that the sides of a triangle are proportional to the sines of the opposite angles, i.e.

$$\frac{a}{\sin \mathrm{A}} = \frac{b}{\sin \mathrm{B}} = \frac{c}{\sin \mathrm{C}}.$$

(4) If two triangles ABC, DEF have $\mathrm{B} = \mathrm{E}$, prove that

$$\frac{\Delta \mathrm{ABC}}{\Delta \mathrm{DEF}} = \frac{\mathrm{AB}\,.\,\mathrm{BC}}{\mathrm{DE}\,.\,\mathrm{EF}}.$$

Example (ii).

To find the area of a regular figure, e.g. a pentagon inscribed in a given circle.

Let O be the centre of the circumscribing circle and AB a side of the figure.

We can find the angle AOB and we thus know two sides and the included angle of the triangle. Five times its area gives the area of the pentagon.

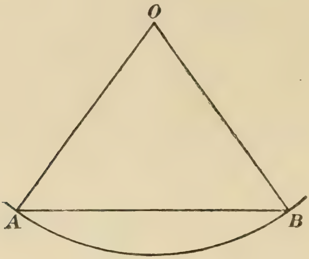

Fig. 25.

Exercise.

(1) Find the area of a regular pentagon inscribed in a circle of radius 10 in.

(2) Find also the perimeter of the pentagon.

(3) Find the area and perimeter of a regular pentagon circumscribed about a circle of 10 in. radius.

Example (iii).

Show that in a triangle $\frac{a}{\sin A}=2R$ where R is the radius of the circumscribing circle.

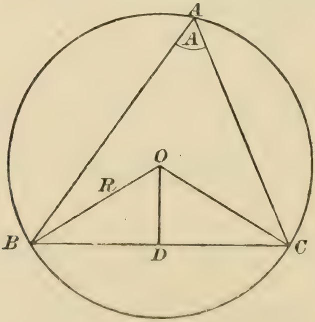

Fig. 26.

Let O be the centre of the circle and D the middle point of BC.

Show that $\angle BOC = 2A$ and hence $\angle BOD = A$. The result easily follows.

Exercise.

(1) Show that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

(2) Prove this also by producing BO to meet the circumference at E and joining EC.

16. *Example* (i).

To an observer on a tower the angles of depression of two points due S. known to be 100 ft. apart are 54° 11′ and 33° 17′. Find the height of the tower above the horizontal plane on which these points lie.

Let x be the required height in feet, AB the tower and C, D the points observed.

Then $$BD = x \cot 33° 17',$$ $$BC = x \cot 54° 11'.$$

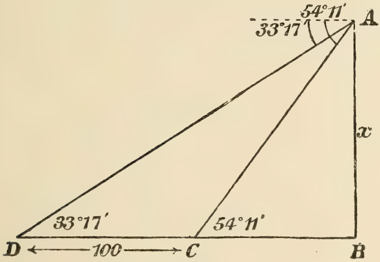

Fig. 27.

$$\therefore\ 100 = x(\cot 33° 17' - \cot 54° 11')$$
$$= x(1{\cdot}5234 - {\cdot}7217)$$
$$= x({\cdot}8017);$$
$$\therefore\ x = \frac{100}{{\cdot}8017} = 125 \text{ ft.}$$

A more convenient method of solving problems of this nature by the aid of logarithms is given in Chap. V. Art. 31.

Example (ii).

To an observer at A the angle of elevation of the top of a tower 220 feet away is 25°, and the angle subtended by the spire above it is 14°. Find the height of the spire.

Let BC represent the tower and CD the spire.

We have $\angle DAB = 39°$ (this is the angle of elevation of the top of the spire).

$DB = 220 \times \tan 39°$,

$CB = 220 \times \tan 25°$;

$\therefore\ CD = 220(\tan 39° - \tan 25°)$

$= 220(0{\cdot}8098 - 0{\cdot}4663)$

$= 220(\cdot 3435)$

$= 75{\cdot}6$ ft.

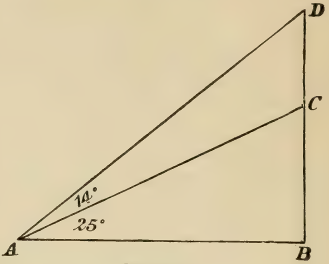

Fig. 28.

17. The Compass.

For purposes of indicating direction the compass is used. In all there are 32 points of the compass, that is, 32 differently named directions from any one point.

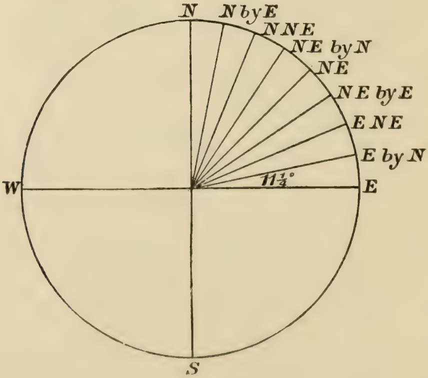

Fig. 29.

Hence the angle between any two consecutive points is $\frac{360^\circ}{32}$, i.e. $11\frac{1}{4}^\circ$.

In the figure we have shown the points in one quadrant. As an Exercise the student should fill in the points in the other quadrants by analogy.

Directions are also often given in degrees. Thus N. 30° E., or 30° East of North, is the direction to the East of North making 30° with the direction North, i.e. between N.N.E. and N.E. by N.

Example.

A man observes a spire in a direction E. 10° N. He walks 500 yards to the S.E. and observes that the bearing of the spire is N.E. How far is he now from the spire?

Let A be his position when he first observes the spire B in the direction AB where $\angle EAB = 10^\circ$.

He walks in the direction AC, 500 yards where $\angle EAC = 45^\circ$. At C the angle $BCN = 45^\circ$ where N is the direction of North.

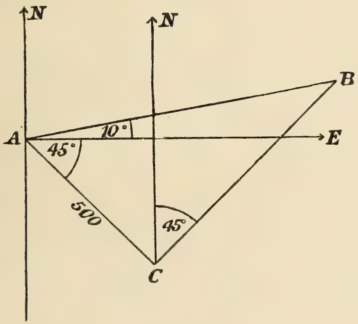

Fig. 30.

$\angle ACB$ being 90° we have

$$\begin{aligned} BC &= 500 \tan 55^\circ \\ &= 714 \text{ yds approx.} \end{aligned}$$

Examples. III c.

1. Find the area of a triangle, given $a=5''$, $b=6''$, $C=43°$.

2. The side of a regular octagon inscribed in a circle is $4''$. Find the radius of the circle.

3. A small weight swings at one end of a string 5 ft. long, the other end being fixed. How far is the weight above its lowest position when the string is inclined at 10° to the vertical?

4. From the top of a cliff 200 feet high the angles of depression of two boats due S. were observed to be 37° and 52°. How far apart were the boats?

5. Find the area and perimeter of a regular hexagon inscribed in a circle of $6''$ radius.

6. From a point A on the ground, the angle of elevation of the top of a tower 60 feet high is 43° 13′. How far is the observer from the foot of the tower and what is the elevation of the tower from a point 10 yards nearer?

7. By how many feet does the shadow cast by a spire 150 ft. high lengthen as the sun sinks from an elevation of 67° 14′ to an elevation of 37° 20′?

8. From a point 8 in. from the centre of a circle of radius 4 in. two tangents are drawn to the circle. Find the angle between them. What is the angle between the radius at the point of contact and the chord of contact? Find the length of the chord of contact.

9. Find the area of a parallelogram whose sides are 4 ft. and 5 ft., the acute angle between them being 47° 17′.

10. A triangle is inscribed in a circle of radius 4·5 cms. with base angles 44° and 56°. Find the lengths of its sides.

11. The sides of a rectangle are $4''$ and $7''$. Find the angle between the diagonals.

12. At a point 100 yards from the foot of a cliff the angle of elevation of the top of the cliff is 35° 11′, and the angle subtended by a tower on its edge is 11° 53′. Find the height of the tower.

13. A man at a point A observes the angle of elevation of the top of a flagstaff to be 35°. He then walks past the flagstaff to a place B on the other side where he observes the angle of elevation to be 63°. From A to B is 120 feet. Find the height of the flagstaff.

14. One side of a triangle inscribed in a circle is 4 in. and the angle opposite it is 27° 11′. Find the diameter of the circle.

15. The road to the top of a hill runs for $\frac{1}{4}$ mile inclined at 10° to the horizon, then for 500 yards at 12°: then for 200 yards at 15°. Find the height of the hill in feet.

Show that the compass directions of the three parts of the road are not required.

16. If a ship sails 4 points off the wind (i.e. in a direction making 45° with the direction of the wind), how far will she have to sail in order to reach a point 30 miles to windward?

17. The shadow of a tower is 55 ft. longer when the sun's elevation is 28° than when it is 42°. Find the height of the tower and the length of the shorter shadow.

18. Find the height of a hill if the angles of elevation taken from two points due North of it and 1000 feet apart are 51° 13′ and 67° 5′.

19. A man in a balloon at a height of 500 ft. observes the angle of depression of a place to be 41°. He ascends vertically and then finds the angle of depression of the same place to be 62°. How far is he now above the ground?

20. A man surveying a mine measures a length AB of 16 chains due E. with a dip of 5° to the horizon; then a length BC of 10 chains due E. with a dip of 3°. How much deeper vertically is C than A? Answer in feet.

21. A building 100 feet long and 50 feet wide has a roof inclined at 35° to the horizon. Find the area of the roof and show that the result will be the same whether the roof has a ridge or not.

22. A man travels 5 miles from A to B in a direction 20° N. of E., then 3 miles to C in a direction N. 25° E. Find the distance of C (1) North of A, (2) East of A, (3) from A. Verify by a figure drawn to scale.

23. The angle of elevation of the top of a house 100 feet high observed from the opposite side of the street is 65°, and the elevation of a window of the house from the same point is 40°. Find the height of the window from the ground.

24. A regular polygon of 10 sides is inscribed in a circle of radius 5 feet. Find the area and perimeter of the polygon and of a circumscribed polygon of the same number of sides.

25. From one end of a viaduct 250 feet long a man observes the angle of depression of a point on the ground beneath to be 37°, and from the other end the angle of depression of this point is 71°. Find the height of the viaduct.

26. The top C of a tower 80 feet high is observed from the top and from the foot of a higher tower AB. From A the angle of depression of C is 18° 11′, and from B the angle of elevation is 23° 31′. Find the height of AB and its distance from the other tower.

27. From a ship the direction of a lighthouse is observed to be N. 25° E., and after the ship has sailed 10 miles North-East, the bearing of the lighthouse is North-West. If the ship now changes her course and sails in direction W. 25° N., how near will she approach the lighthouse?

28. A man standing at a point A on the bank of a river wishes to find the distance of a point B directly opposite him on the other bank. He noticed a point C also on the other bank and found $\angle BAC$ to be 55°; he walked directly away from the river for 100 yards to a point D and found the angle ADC to be 35°. Find the distance AB.

29. From a steamer moving in a straight line with a uniform velocity of 10 miles per hour the direction of a lighthouse is observed to be N.W. at midnight, W. at 1 a.m., S. at 3 a.m. Show that the direction of the steamer's course makes an angle $\cot^{-1} 3$ with the N. Find the least distance of the steamer from the lighthouse.

30. B is 50 yards from A in a direction E. 20° S., C is 100 yards from B in a direction E. 32° 15′ N., D is 80 yards from C in a direction W. 46° 10′ N. Find how far D is from A and in what direction.

Miscellaneous Examples. B.

1. Draw two straight lines OB, OC at right angles and OA between them making 39° with OB. With centre O and radius 10 cms. draw a circle cutting OB in Q and OA in P. From P let fall perpendiculars PS on OB and PR on OC. At Q draw a tangent QT cutting OA in T. Measure PR, PS, QT to the nearest millimetre and write down their lengths. Hence find sin 39°, cos 39°, tan 39° and compare with the values given in the tables.

2. The diagonal of a rectangle is 12 cms. long and makes an angle of 34° with one of the sides. Find the length of the sides.

3. Prove that $(\sin A+\cos A)^2=1+2\sin A\cos A$; and hence evaluate $\sqrt{1+2\sin 53^\circ\cos 53^\circ}$.

4. Find the values of

(i) $\sin 47^\circ \sec 47^\circ$;

(ii) $\tan 74^\circ \operatorname{cosec} 74^\circ$.

5. The base of an isosceles triangle is 8 cms. and the diameter of its circumscribing circle is 12 cms. Find its vertical angle and its altitude.

---

6. AB is a diameter of a circle, centre O, and OC is a radius. If $OC=a$ and $\angle COB=\alpha$, show that $AC=2a\cos\frac{\alpha}{2}$ and the length of the perpendicular from O on $AC=a\sin\frac{\alpha}{2}$.

7. Draw accurately a triangle with base $BC=5$ cms., $BA=8$ cms., $B=40°$. Calculate the length of the perpendicular from A on BC. Find the area of the triangle and from measurements of your diagram find cos 40°.

8. A man 5 ft. 9 in. high standing 134·2 ft. from the foot of a tower observes the elevation of the tower to be 30° 14′. Find the height of the tower.

9. Prove that if $\cos A=a$ then $\tan A=\dfrac{\sqrt{1-a^2}}{a}$.

10. P, Q, R are three villages. P lies 7 miles to the N.E. of Q and Q lies $11\frac{1}{4}$ miles to the N.W. of R. Find the distance and bearing of P from R.

---

11. Two adjacent sides of a parallelogram are AB = 6 cms., BC = 7 cms., the included angle being 52°. Find the angles between the diagonal BD and the sides AB and BC. Verify by an accurate drawing.

12. A ladder 20 ft. long rests against a vertical wall and makes an angle of 50° with the ground. What will be its inclination to the ground when the foot of the ladder is 5 ft. farther from the wall?

13. Express the equation $2\cos^2\theta + \sin\theta = 2$, in terms of $\sin\theta$, solve it and find from the tables two values of θ to satisfy it.

14. Two equal forces P making an angle a with one another act at a point O. Their resultant R is represented by the diagonal passing through O of the parallelogram in which the lines representing the forces form two adjacent sides. Prove

$$R = 2P\cos\frac{a}{2}.$$

15. Show from a figure that $\cot 53° = \tan 37°$ and hence find a value of θ which satisfies the equation $\cot(\theta + 16°) = \tan\theta$.

---

16. In a triangle ABC, $a = 2''$, $c = 3''$, $B = 37°$; calculate the length of the perpendicular drawn from A to BC. Also if PBC be an isosceles triangle on BC as base and of the same altitude as the triangle ABC, find its angles.

17. Express $16\sin\theta + 3\operatorname{cosec}\theta = 16$ as a quadratic in $\sin\theta$ and find two values of θ to satisfy it.

18. On a tower 85 ft. high stands a pole of length 10 ft. What angle does this pole subtend at a point on the horizontal plane on which the tower stands, at a point 40 ft. from its base?

19. Find the area of a regular pentagon inscribed in a circle of 4″ radius.

20. O the mid-point of AC is the centre of the circle circumscribing the right-angled triangle ABC. If $b = 13$, $c = 12$, find a. Prove that $\angle BOC = 2A$. Find $\sin 2A$, $\sin A$, $\cos A$, and verify the relation $\sin 2A = 2\sin A\cos A$.

21. A man at a point B observes an object at C and walks 200 yards in a direction making an angle of 68° with BC, to a point A where the angle CAB also equals 68°. Find the distance from B to C.

22. A set square has its hypotenuse 12″ long and the shorter side 4″. The hypotenuse slides along a scale which is held fixed, and an arrowhead on the hypotenuse is placed in succession against marks at intervals of 0·15 of an inch on the scale. In each position a line is ruled along the longer side of the set square. How far apart are these lines? If an error of 0·01 of an inch was made in placing the set square, what error in the position of the line would result?

23. If a ship after sailing 25 miles is 12 miles to windward of her starting point, what angle does her course make with the direction of the wind?

24. Construct the angle whose cotangent is 1·62. Measure it and compare with the angle given in the tables.

25. Find two values of θ to satisfy the equation

$$12 \cot \theta + 6 \tan \theta = 17.$$

---

26. In the side of a hill which slopes at an angle of 20° to the horizontal, a tunnel is bored sloping downwards at an angle of 10° with the horizontal. How far is a point 40 ft. along the tunnel vertically below the surface of the hill?

27. Find two values of θ to satisfy the equation

$$6 \cos^2 \theta + 7 \sin \theta - 8 = 0.$$

28. A surveyor finds two points A, B on a hillside to be 3 chains 43 links apart, and finds the line AB to be inclined at 17° 30′ to the horizontal. On his plan these points must be shown at their horizontal distance apart. What is this to the nearest link? Given 1 chain = 100 links.

29. If $x = a \sec \theta$, $y = b \tan \theta$, prove that $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

30. C is the right angle of a right-angled triangle ABC. AD and BD are drawn perpendicular to AC and AB respectively. Prove that AD = BC cosec^2 A.

CHAPTER IV.

FUNCTIONS OF ANGLES GREATER THAN A RIGHT ANGLE.

18. Note on the Convention of Sign.

If a line OX of indefinite length be drawn from a point O, and any length such as OM be taken as a unit, we may

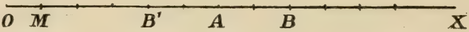

Fig. 31.

represent any integral number by the length of a segment containing this number of units, e.g. OA, which contains OM six times, represents the number 6, and AB the number 2.

If we wish to add the two numbers represented by OA, AB we may place AB at the end of OA and we have their sum represented by OB.

If we wish to subtract AB from OA we have only to mark off AB′ equal to AB but in the opposite direction, and we have OB′ their difference.

If AB is longer than OA, B′ falls to the left of O and the difference is represented by OB′, measured from O from *right to left* and not from left to right.

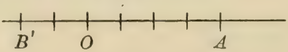

Fig. 32.

Thus if OA $= 4$ and AB $= 6$. Mark off OA from O, four divisions to the right, and then from A, six divisions to the left. OB′ represents $4 - 6 = -2$.

It will thus be seen that lengths measured along a line

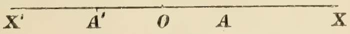

Fig. 33.

XX′ from a point O will be conveniently regarded as positive if taken in the direction OA to the right of O but as negative if drawn to the left.

This difference in sign may also be represented by the order of the letters; thus OA may be considered as −AO.

OA and AO are said to denote the same segment taken in opposite *senses*.

Similarly, for lengths measured along a line OY at right

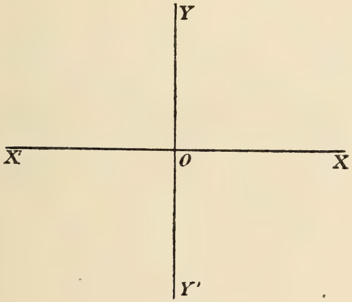

Fig. 34.

angles to XX′, the direction OY is considered positive and OY′ negative.

This convention is applied, not only to lengths measured along XX′ and YY′, but also to lines drawn parallel to these.

It will be found that, by the adoption of these conventions, trigonometrical formulae are considerably simplified and that instead of requiring different formulae for cases in which the angle involved is acute or obtuse, positive or negative, we are able to use the same formula for all cases.

19. As in Article 1 we will suppose a straight line, called the *radius vector*, to turn about O from an initial position OX; then the amount of revolution it has undergone in coming to the final position OP measures the angle XOP. Also it will be remembered that if PN be the perpendicular drawn from P to the initial line OX, then

$$\sin XOP = \frac{NP}{OP}, \quad \cos XOP = \frac{ON}{OP}, \quad \tan XOP = \frac{NP}{ON},$$

whatever position OP may have.

It is important to notice that all angles are supposed to be described by revolution *from the position* OX.

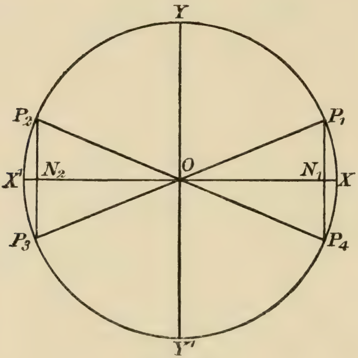

Fig. 35.

This revolution may be in the opposite direction to that of the hands of a clock, called the positive direction; or in the same direction as the hands of a clock, called the negative direction.

Also the radius vector may make any number of complete revolutions before coming to rest.

From our definition it follows that all angles which have the same boundary line OP have the same trigonometrical functions. Such angles are called *coterminal angles.*

20. In the figure, Art. 19, YOY' is drawn perpendicular to XOX', so that any circle described with O as centre is divided into four quadrants. The quadrants XOY, YOX', $X'OY'$, $Y'OX$ are called the first, second, third, and fourth quadrants respectively.

Now if the lines P_1OP_3, P_2OP_4 are equally inclined to XX', we have four congruent triangles

$$P_1ON_1, \quad P_2ON_2, \quad P_3ON_2, \quad P_4ON_1.$$

Hence it follows that the trigonometrical functions of the angles XOP_1, XOP_2, XOP_3, XOP_4 are *numerically* the same; also that there are *four and only four* positions which the boundary line may have in order that any one trigonometrical function of the angle may have a given *numerical* value.

If θ be the acute angle XOP_1 we see from the figure that

$$\sin\theta, \quad \sin(180^\circ-\theta), \quad \sin(180^\circ+\theta), \quad \sin(360^\circ-\theta)$$

are *numerically* equal; and so for the other functions.

Here it is convenient to adopt the convention of sign which we have mentioned already.

The convention of sign is as follows:

The radius vector OP is always considered positive.

ON is considered positive if measured along OX, and negative if measured along OX'.

NP is considered positive if measured in the direction OY, and negative if measured in the direction OY'.

Hence if OP lies in the first quadrant,

ON and NP are positive;

$\therefore$ all the functions are positive.

If OP lies in the second quadrant,

ON is negative, and NP is positive;

$\therefore$ the sine and cosecant are positive, but all the other functions are negative.

If OP lies in the third quadrant,

ON and NP are negative;

∴ the tangent and cotangent are positive, but all the other functions are negative.

If OP lies in the fourth quadrant,

ON is positive, and NP is negative;

∴ the cosine and secant are positive, but all the other functions are negative.

Thus if in the figure of Art. 19, $PN : ON : OP = 3 : 4 : 5$,

$$\sin XOP_1 = \sin XOP_2 = \tfrac{3}{5}, \quad \sin XOP_3 = \sin XOP_4 = -\tfrac{3}{5};$$

$$\cos XOP_1 = \cos XOP_4 = \tfrac{4}{5}, \quad \cos XOP_2 = \cos XOP_3 = -\tfrac{4}{5};$$

$$\tan XOP_1 = \tan XOP_3 = \tfrac{3}{4}, \quad \tan XOP_2 = \tan XOP_4 = -\tfrac{3}{4}.$$

Now, having regard to the *sign* of the function, we see that there are *two* positions which the boundary line may have when we are given the value of any one function.

21. The point of chief importance for us is that we may be able to obtain at once any trigonometrical function of any angle α with the help of tables which give the functions of acute angles only.

The most convenient method is to notice in which quadrant the boundary line of α lies, and then to obtain from the tables the required functions of the acute angle θ, where

$$\alpha = 180° - \theta \text{ for the second quadrant,}$$

$$\alpha = 180° + \theta \text{ for the third quadrant,}$$

$$\alpha = 360° - \theta \text{ for the fourth quadrant.}$$

We then only have to prefix the proper sign, which can be done by drawing a figure, or mentally after a little practice.

Example (i).

Find the functions of 140°.

The boundary line of the angle is in the second quadrant, and the corresponding acute angle is 40°, since $140° = 180° - 40°$. Also in the second quadrant the sine is positive, and the cosine and tangent are negative;

$$\therefore \sin 140° = \sin 40° = \cdot 6428;$$
$$\cos 140° = -\cos 40° = -\cdot 7660;$$
$$\tan 140° = -\tan 40° = -\cdot 8391.$$

In a similar way we have

$$\cos 200° = \cos(180° + 20°) = -\cos 20° = -\cdot 9397;$$
$$\tan 313° = \tan(360° - 47°) = -\tan 47° = -1\cdot 0724;$$
$$\operatorname{cosec} 127° = \operatorname{cosec}(180° - 53°) = +\operatorname{cosec} 53° = 1\cdot 2521;$$
$$\cot 197°\,24' = \cot(180° + 17°\,24') = +\cot 17°\,24' = 3\cdot 1910.$$

Example (ii).

Find the positive angles less than 360° which satisfy

$$(1)\ \tan\theta = \cdot 4734;\quad (2)\ \cos\theta = -\cdot 4360.$$

(1) Since the tangent is positive the boundary lines of the angles must be in the first and third quadrants.

From the tables, $\cdot 4734 = \tan 25°\,20'$;

$\therefore$ the angle in the first quadrant is $25°\,20'$; and the angle in the third quadrant is

$$180° + 25°\,20', \text{ i.e. } 205°\,20'.$$

(2) Since the cosine is negative the boundary lines of the angles must be in the second and third quadrants.

From the tables, $\cdot 4360 = \cos 64°\,9'$;

$\therefore$ the angle in the second quadrant is

$$180° - 64°\,9', \text{ i.e. } 115°\,51';$$

and the angle in the third quadrant is

$$180° + 64°\,9', \text{ i.e. } 244°\,9'.$$

22. If we are given $\sin\theta = \frac{3}{5}$, we have

$$\cos^2\theta = 1 - \sin^2\theta = \tfrac{16}{25};$$

$$\therefore \quad \cos\theta = \pm\tfrac{4}{5}.$$

The meaning of the double sign, which we disregarded in Art. 11, can now be explained.

There are two positions which the boundary line of θ may have in order that $\sin\theta$ may be $\frac{3}{5}$, one in the first quadrant and one in the second.

The cosines of angles which have one of these two boundary lines are numerically $\frac{4}{5}$; but if the boundary line is in the first quadrant the cosine is positive, and if the boundary line is in the second quadrant the cosine is negative.

23. We can state our results more generally as follows:

$$\sin(180° - \theta) = \sin\theta,$$

$$\cos(180° - \theta) = -\cos\theta,$$

$$\tan(180° - \theta) = -\tan\theta,$$

$$\sin(180° + \theta) = -\sin\theta,$$

$$\cos(180° + \theta) = -\cos\theta,$$

$$\tan(180° + \theta) = \tan\theta.$$

Also since the angles $-\theta$, $360° - \theta$ are coterminal, we have

$$\sin(-\theta) = \sin(360° - \theta) = -\sin\theta,$$

$$\cos(-\theta) = \cos(360° - \theta) = \cos\theta.$$

The student who wishes to acquire skill in trigonometrical transformations should make himself familiar with the results in the above form, and with the functions of $90° - \theta$ and $90° + \theta$ which we discuss in Articles 24, 25.

24. To prove that

$$\sin(90° - \theta) = \cos\theta,$$

$$\text{and}\ \cos(90° - \theta) = \sin\theta.$$

Let a radius vector start from OX and revolve until it has described an angle θ, taking up the position OP.

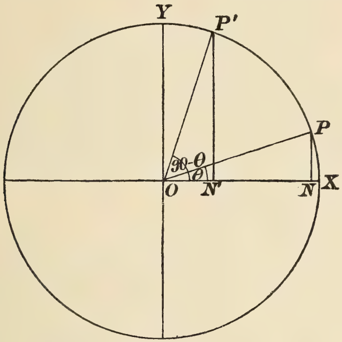

Fig. 36.

Then let the radius vector start from OX and revolve through $90°$ to the position OY and back through an angle θ to the position OP′. Then XOP′ is the angle $90° - \theta$.

If we draw perpendiculars PN, P′N′ to OX, we have two congruent triangles PON, ON′P′.

Hence $\quad \sin(90° - \theta) = \dfrac{N'P'}{OP'} = \dfrac{ON}{OP} = \cos\theta,$

$$\cos(90° - \theta) = \frac{ON'}{OP'} = \frac{NP}{OP} = \sin\theta.$$

Thus we have important relations between the functions of complementary angles.

The sine of an angle is the cosine of its complement.

The tangent of an angle is the cotangent of its complement.

The secant of an angle is the cosecant of its complement.

Example.

Find a value of θ to satisfy $\sin 6\theta = \cos 4\theta$.

The equation is satisfied if 6θ and 4θ are complementary angles; that is if $6\theta + 4\theta = 90°$; hence $\theta = 9°$ is a solution of the equation.

25. To prove that

$$\sin(90° + \theta) = \cos\theta,$$
$$\text{and } \cos(90° + \theta) = -\sin\theta.$$

Let a radius vector start from OX and revolve until it has described an angle θ, taking up the position OP.

Then let the radius vector start from OX and revolve through 90° to the position OY and then on through the angle θ to the position OP′.

Then XOP′ is the angle $90° + \theta$.

If we draw the perpendiculars PN, P′N′ to OX, we have two congruent triangles PON, ON′P′; and hence

ON′ = NP in magnitude but is of opposite sign,

N′P′ = ON in magnitude and is of the same sign;

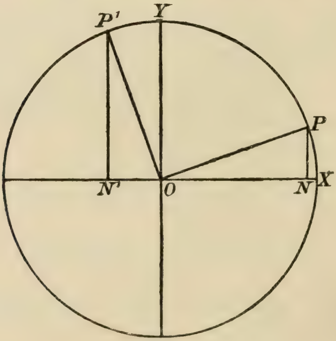

Fig. 37.

$$\therefore \sin(90° + \theta) = \frac{N'P'}{OP'} = \frac{ON}{OP} = \cos\theta,$$

$$\cos(90° + \theta) = \frac{ON'}{OP'} = -\frac{NP}{OP} = -\sin\theta.$$

26. The results of Articles 23, 24, 25 have been obtained for the case in which θ is an acute angle, but the importance of the sign convention will be realised by noticing that we obtain the same formulae whatever the magnitude of θ may be. The following figures illustrate the relations between the ratios of θ and $180° - \theta$.

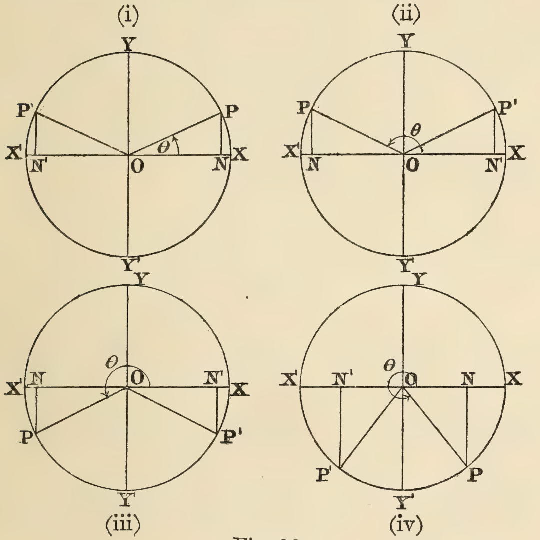

Fig. 38.

In each figure XOP represents the angle θ, taken in turn, in each of the four quadrants.

XOP′ represents the angle $180° - \theta$ formed by OP′ turning from OX through 180° and then backwards in the negative direction through an angle equal to θ.

It will be seen that in each case $NP = N'P'$, $ON = -ON'$ and OP is always considered positive;

$$\therefore \sin(180° - \theta) = \sin\theta, \quad \cos(180° - \theta) = -\cos\theta,$$
$$\tan(180° - \theta) = -\tan\theta.$$

Exercise. Draw figures to prove the relations between the ratios of the angles θ, $180° + \theta$, $90° - \theta$, $90° + \theta$ when

(i) $\theta = 150°$, (ii) $\theta = 215°$, (iii) $\theta = -30°$.

Examples. IV a.

1. With the help of the tables, find the following:

(1) $\sin 115^\circ$.	(2) $\cos 130^\circ$.	(3) $\sec 175^\circ$.
(4) $\tan 142^\circ$.	(5) $\cos 312^\circ$.	(6) $\cot 127^\circ$.
(7) $\sin 125^\circ 37'$.	(8) $\cos 98^\circ 14'$.	(9) $\sin 216^\circ$.
(10) $\tan 243^\circ 15'$.	(11) $\operatorname{cosec} 164^\circ$.	(12) $\cot 192^\circ 33'$.

2. Find in each of the following cases two positive values of θ less than 360°:

(1) $\tan\theta = -2{\cdot}1426$. (2) $\tan\theta = {\cdot}3466$. (3) $\sin\theta = {\cdot}8916$.
(4) $\cos\theta = -{\cdot}3870$. (5) $\operatorname{cosec}\theta = -1{\cdot}1432$. (6) $\cot\theta = 2{\cdot}9515$.

3. Draw the boundary lines of all the angles whose tangent is $\cdot 7$. Measure the two smallest positive angles with a protractor, and verify your results with the tables.

4. Draw a figure to show that if $\sin\theta = \frac{5}{13}$, then $\tan\theta = \pm\frac{5}{12}$.

5. When $A = 130^\circ$, draw figures to show that

$$\sin(90^\circ + A) = \cos A, \quad \sin(180^\circ + A) = -\sin A,$$
$$\tan(180^\circ - A) = -\tan A.$$

6. In a triangle ABC, $b = 5$, $c = 3$; show that the area is the same whether $A = 50^\circ$ or 130°.

7. If A is an angle of a triangle, find its magnitude from the following equations:

$$(1)\ 3\sin A = 1{\cdot}7;\quad (2)\ 4\cos A = 2{\cdot}5;\quad (3)\ 5\cos A + 2 = 0.$$

8. Show that no root of $5\sin\theta + 4 = 0$ can be an angle of a triangle.

9. Find all the positive angles between 0° and 360° which satisfy the equations

$$(1)\ 2\cos^2\theta = 3\sin\theta;\quad (2)\ 10\sin^2\theta - 3\sin\theta - 4 = 0;$$
$$(3)\ 10\tan\theta - 5\cot\theta = 23.$$

10. By making use of the relations which exist between the functions of complementary angles, find a value of θ to satisfy the equations

$$(1)\quad \sin 3\theta = \cos 2\theta;\quad (2)\quad \tan 5\theta = \cot 4\theta.$$

11. By using the relations which exist between functions of θ and $180^\circ \pm \theta$, find a value of θ to satisfy the equations

$$(1)\ \sin 4\theta = \sin\theta;\quad (2)\ \sin 4\theta = -\sin\theta;\quad (3)\ \cos 4\theta = -\cos\theta.$$

12. If A, B, C are the angles of a triangle, show that

$$(1)\quad \sin(B + C) = \sin A;\qquad (2)\quad \cos(B + C) = -\cos A;$$
$$(3)\qquad \sin\frac{A + B}{2} = \cos\frac{C}{2}.$$

27. Limiting Values.

If the denominator of a fraction remains constant while the numerator decreases, it is clear that the fraction decreases; and by decreasing the numerator sufficiently the fraction can be made as small as we please.

Thus in the fraction $\frac{x}{a}$, if a remains constant while x decreases, the fraction also decreases and approaches zero; that is to say, its numerical value becomes and remains less than any positive number we may choose, no matter how small. A convenient notation to express this is

$$\underset{x \to 0}{\mathrm{L}} \frac{x}{a} = 0.$$

This is read "as x tends to zero, the limiting value of the fraction $\frac{x}{a}$ is zero."

If the numerator of a fraction remains constant while the denominator decreases, the fraction increases.

e.g. $\quad \frac{a}{\cdot 1} = 10a$; $\frac{a}{\cdot 001} = 1000a$; $\frac{a}{\cdot 000001} = 1000000a$.

By making the denominator sufficiently small we can make this fraction become and remain greater than any positive number we may choose, however great that number may be; in this case the fraction tends to infinity

$$\left(\frac{a}{x} \to \infty\right).$$

A convenient notation to express this is

$$\underset{x \to 0}{\mathrm{L}} \frac{a}{x} = \infty.$$

Similarly it will be seen that

$$\underset{x \to \infty}{\mathrm{L}} \frac{a}{x} = 0.$$

28. Functions of 0° and 90°.

Let XOP be any small angle.

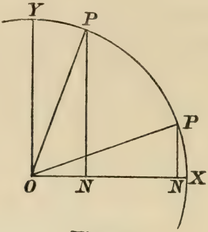

Fig. 39.

With our usual notation we have

$$\sin \text{XOP} = \frac{\text{NP}}{\text{OP}}.$$

Now as the radius vector approaches the position OX, NP decreases while OP remains constant.

Hence as the angle XOP decreases, sin XOP also decreases; and in the limit, when OP lies along OX, we have

$$\sin 0^\circ = \frac{0}{\text{OP}} = 0.$$

Also as OP approaches OX, ON becomes more nearly equal to OP, and in the limit we have

$$\cos 0^\circ = \frac{\text{OP}}{\text{OP}} = 1.$$

Again, if we consider the ratio $\frac{\text{OP}}{\text{NP}}$, we see that as the angle XOP decreases OP remains constant while NP decreases; and therefore the ratio $\frac{\text{OP}}{\text{NP}}$ increases. In the limit when NP vanishes, the ratio becomes infinitely great; and hence we have

$$\text{cosec } 0^\circ = \infty.$$

In a similar way it can be shown that

$$\tan 0^\circ = 0, \quad \cot 0^\circ = \infty, \quad \sec 0^\circ = 1.$$

Now let us suppose OP to approach the line OY. In this case NP approaches OP and coincides with it when

$$\angle XOP = 90°,$$

and ON decreases and becomes zero when

$$\angle XOP = 90°.$$

Hence
$$\sin 90° = \frac{OP}{OP} = 1,$$

$$\cos 90° = \frac{0}{OP} = 0,$$

$$\tan 90° = \frac{OP}{0} = \infty.$$

When the angle XOP becomes slightly greater than 90°, ON becomes negative and the tangent of the angle is infinitely great and of negative sign. The tangent is said to change its sign when passing through the value infinity.

It will be noticed that 0° and 90° are complementary angles and consequently their functions obey the laws of Art. 24.

Exercise. Write down the values of :

(i) cosec 90°, sec 90°, cot 90°.

(ii) sin 180°, cos 180°, tan 180°.

(iii) cosec 180°, sec 180°, cot 180°.

(iv) sin 270°, cos 270°, tan 270°.

(v) cosec 270°, sec 270°, cot 270°.

29. To trace the changes in the functions as the angle changes from 0° to 360°.

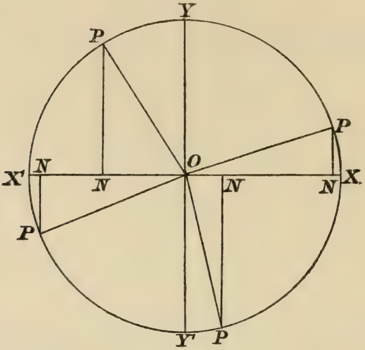

Fig. 40.

With the same figure as before, let $\angle XOP = \theta$.

Then $$\sin\theta = \frac{NP}{OP}.$$

Now OP remains constant in magnitude and sign, so the changes in $\sin\theta$ are due to the changes in NP only.

When $\theta = 0°$ we have $\sin\theta = 0$ [Art. 28].

As θ increases from 0° to 90°,

NP increases and is positive;

$\therefore$ $\sin\theta$ increases and is positive.

When $\theta = 90°$, $\sin\theta = 1$ [Art. 28].

As θ increases from 90° to 180°,

NP decreases and is positive;

$\therefore$ $\sin\theta$ decreases and is positive.

When $\theta = 180°$, $\sin\theta = \frac{0}{OP} = 0.$

As θ increases from 180° to 270°,

NP increases and is negative;

$\therefore$ $\sin\theta$ increases and is negative.

When $\theta = 270°$, $\sin\theta = -\dfrac{\text{OP}}{\text{OP}} = -1.$

As θ increases from 270° to 360°,

NP decreases and is negative;

$\therefore$ $\sin\theta$ decreases and is negative.

When $\theta = 360°$, $\sin\theta = \dfrac{0}{\text{OP}} = 0.$

The changes in the value of a function can be shown conveniently by means of a curve drawn on squared paper.

Draw two axes OX, OY at right angles to one another. Along OX take a length ON to represent the magnitude of an angle, and erect a perpendicular NP to represent the value of the function. The locus of P will be a curve which is called the graph of the function.

We have given below the graph of the sine.

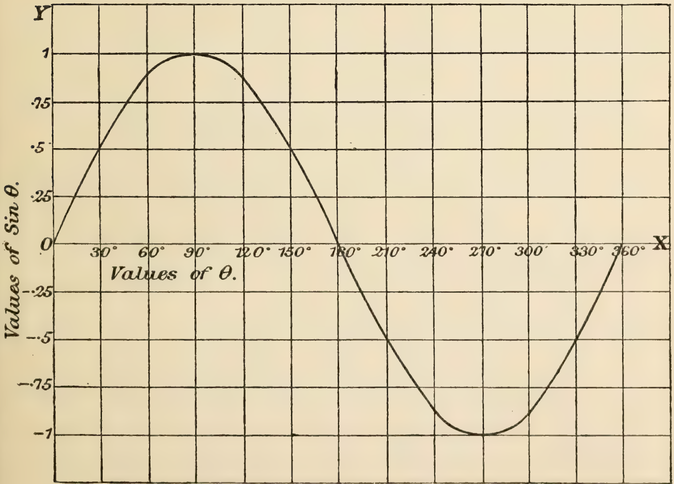

Fig. 41.

30. To trace the changes in $\tan\theta$, we have $\tan\theta = \frac{\text{NP}}{\text{ON}}$, and both NP and ON change with θ.

When $\theta = 0°$, $\tan\theta = \frac{0}{\text{OP}} = 0$.

As θ changes from $0°$ to $90°$,

NP increases and is positive,

ON decreases and is positive;

$\therefore$ $\tan\theta$ increases and is positive.

When $\theta = 90°$, $\tan\theta = \frac{\text{OP}}{0} = \infty$.

As θ changes from $90°$ to $180°$,

NP decreases and is positive,

ON increases and is negative;

$\therefore$ $\tan\theta$ decreases and is negative.

When $\theta = 180°$, $\tan\theta = \frac{0}{\text{OP}} = 0$.

As θ changes from $180°$ to $270°$,

NP increases and is negative,

ON decreases and is negative;

$\therefore$ $\tan\theta$ increases and is positive.

When $\theta = 270°$, $\tan\theta = \frac{\text{OP}}{0} = \infty$.

As θ changes from $270°$ to $360°$,

NP decreases and is negative,

ON increases and is positive;

$\therefore$ $\tan\theta$ decreases and is negative.

When $\theta = 360°$, $\tan\theta = \frac{0}{\text{OP}} = 0$.

The graph of $\tan\theta$ is given below. Note that since

$$\tan(180^\circ+\theta)=\tan\theta,$$

the curve for values of θ from 0° to 180° is repeated for values of θ from 180° to 360°.

In drawing graphs of the functions the student should note that the function changes sign only after passing through the values zero or infinity.

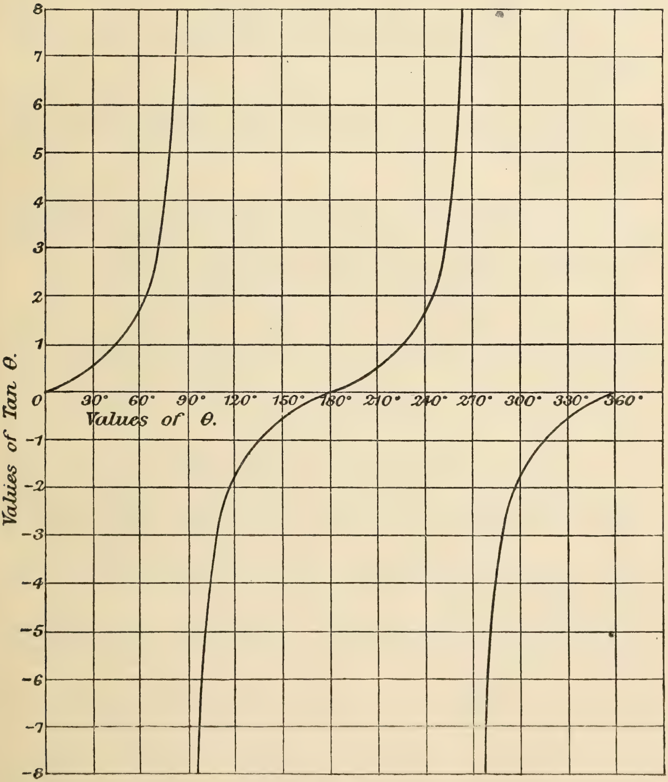

Fig. 42.

Examples. IV b.

1. Discuss the changes in the following functions as θ changes from $0°$ to $360°$, and illustrate by a graph in each case:

(1) $\cos\theta$. (2) $\cot\theta$. (3) $\operatorname{cosec}\theta$. (4) $\sec\theta$.

2. Draw, with the same axes of reference, graphs of $\sin\theta$ and $\cos\theta$; and from your figure obtain values of θ between $0°$ and $360°$ for which (1) $\sin\theta=\cos\theta$; (2) $\sin\theta=-\cos\theta$.

Also with the help of your figure draw the graph of $\sin\theta+\cos\theta$.

3. Trace the changes in sign and magnitude of $\tan\theta$ as θ decreases from $180°$ to $0°$.

4. Draw a curve on squared paper to show the length of the shadow cast by a tree 100 ft. high for all elevations of the sun up to $50°$.

5. N is the foot of the perpendicular drawn from a moving point P to the fixed straight line OX. If all positions of P are obtained by giving different values to θ in the equations

$$\mathrm{ON}=5\cos\theta, \qquad \mathrm{PN}=4\sin\theta,$$

find for what values of θ, P is (1) nearest to O, (2) farthest from O, and obtain the distance of P from O in each case.

Draw on squared paper a curve showing the positions of P for values of θ between $0°$ and $180°$.

6. A particle projected with a velocity of 100 feet per second in a direction making an angle a with the horizontal plane strikes the horizontal plane again at a distance $\dfrac{10000\sin 2a}{32}$ ft. from the point of projection. For what value of a is this distance greatest, and what is the greatest distance?

Also find two values of a for which the range of the particle would be 100 ft.

CHAPTER V.

RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

31. To prove that in any triangle

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

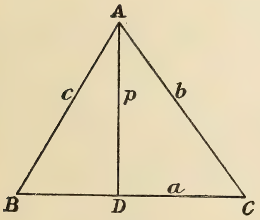

Fig. 43.

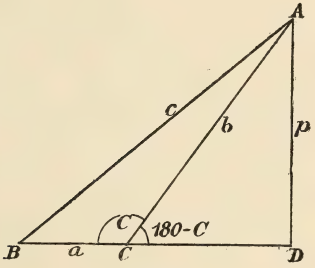

Fig. 44.

If p be the length of the perpendicular AD drawn from A to the side BC, we have

$$p = c \sin B = b \sin C;$$

$$\therefore \quad \frac{b}{\sin B} = \frac{c}{\sin C};$$

and in a similar way each of these ratios may be shown to be equal to $\frac{a}{\sin A}$.

If one of the angles be obtuse, such as C in Fig. 44, the same result holds, for $p = b \sin (180^\circ - C) = b \sin C$ as before.

Note. Prove that the formula $\frac{1}{2}ab \sin C$ for the area of a triangle (Art. 15), holds good when C is an obtuse angle.

32. To prove that in any triangle

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

In Fig. 43, where C is acute, we have, by a well-known theorem in Geometry,

$$\begin{aligned} AB^2 &= BC^2 + CA^2 - 2BC \,.\, CD \\ &= BC^2 + CA^2 - 2BC \,.\, CA\cos C. \end{aligned}$$

i.e. $c^2 = a^2 + b^2 - 2ab\cos C.$

In Fig. 44, where C is obtuse, we have, by Geometry,

$$\begin{aligned} AB^2 &= BC^2 + CA^2 + 2BC \,.\, CD \\ &= BC^2 + CA^2 + 2BC \,.\, CA\cos(180^\circ - C) \\ &= BC^2 + CA^2 - 2BC \,.\, CA\cos C, \end{aligned}$$

i.e. $c^2 = a^2 + b^2 - 2ab\cos C.$

Note that the sign convention enables us to have one formula for both cases, (i) C acute, (ii) C obtuse.

Thus we can obtain the third side of a triangle when we are given two sides and the included angle.

And since the above formula can be written

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

and similarly

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca},$$

we can obtain any angle of a triangle of which the sides are known.

33. When any three *independent* parts of a triangle are given, the formulae proved above are sufficient to determine the remaining parts, but the complete solution of triangles without the use of logarithms generally involves clumsy work, and we shall therefore postpone it until Chapter VIII.

There are however many occasions on which logarithmic work is not required, and it is well that the student should become familiar with the formulae at this stage.

Example (i).

Find the largest angle of the triangle whose sides are 3, 4, 6. If $a=3$, $b=4$, $c=6$, we have

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{9+16-36}{24} = -\frac{11}{24} = -\cdot 4583;$$

$$\therefore \quad C = 180° - 62° 43'$$
$$= 117° 17'.$$

Example (ii).

A man observes the elevation of a tower to be a; after walking a distance c towards the tower he observes the elevation to be β. Find the height of the tower.

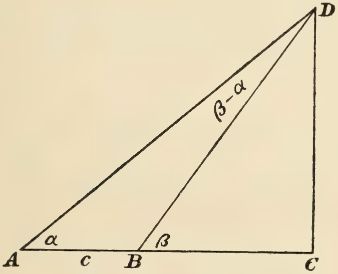

Fig. 45.

Let A, B denote the points of observation, and CD the tower.

Then $$CD = BD \sin \beta.$$

Now from the $\triangle$ ABD, we have

$$\frac{BD}{\sin a} = \frac{c}{\sin(\beta - a)};$$

$$\therefore \quad BD = \frac{c \sin a}{\sin(\beta - a)},$$

and the height of the tower is

$$\frac{c \sin a \sin \beta}{\sin(\beta - a)}.$$

This method is usually more convenient than that given in Article 16, as the result is suitable for logarithmic work.

34. If we are given two sides of a triangle and the angle opposite one of them, say a, b, A, we may proceed to find the remaining parts in two ways.

We may find the angle B from the relation

$$\sin B = \frac{b \sin A}{a} \quad \text{.....................(1),}$$

or we may find the side c by considering the relation

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{...............(2)}$$

as a quadratic equation in c.

Now from (1) we get two values for B, which are supplementary angles [Art. 23], and from the quadratic equation (2) we get two values for c.

There may consequently be ambiguity concerning the solution of the triangle, which we will now discuss geometrically.

35. To construct the triangle, draw the angle XAC equal to A, and make AC equal to b. With centre C and radius a describe a circle, which (if the data are possible) will meet AX at the required point B.

If $a =$ the perpendicular drawn from C to AX $= b \sin A$, the circle touches AX at B [see Fig. 46].

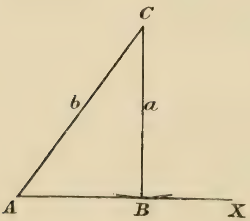

Fig. 46.

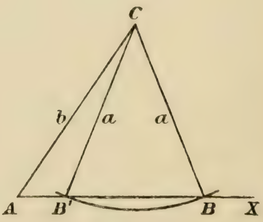

Fig. 47.

If $a > b \sin A < b$, the circle cuts AX at two points B, B′, and we have ambiguity; for both triangles CAB, CAB′ have the given parts [see Fig. 47].

If $a=b$, the point B′ coincides with A, and we have one triangle only.

If $a>b$, the points B, B′ are on opposite sides of A, and we only have one triangle with the given parts; for ∠ CAB′ is the supplement of the given angle A [see Fig. 48].

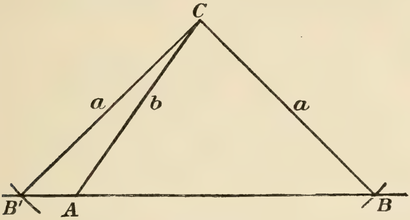

Fig. 48.

Thus we see that ambiguity can only arise when the side opposite the given angle is less than the other side.

Examples. V a.

1. Find the largest angle of the triangle whose sides are 6, 7, 8 feet.

2. Given $B=114°$, $a=2$, $c=3$, find b.

3. Find the vertical angle of an isosceles triangle whose equal sides are 3 ft. and base 5 ft.; (1) by using the fact that the bisector of the vertical angle is perpendicular to the base and bisects it; (2) by using the formula giving the cosine of an angle of a triangle in terms of the sides.

4. Find the lengths of the diagonals of a parallelogram of which two sides are 2, 5 metres and are inclined at 50°.

5. Show that the parts $B=40°$, $b=5$, $c=20$ cannot form a triangle.

6. If $a=4$, $b=5$, $c=6$, find the angles.

7. The diagonals of a parallelogram are 4, 6 ft. and intersect at 28°; find the sides.

8. If $b=10$, $c=8$, $A=47°$, solve the triangle.

9. Show that there are two triangles having $b=3$, $c=4$, $B=40°$, and find the angle A in each case.

10. The sides of a parallelogram are 4 ft., 5 ft., and the shorter diagonal is 2 ft.; find the other diagonal.

11. Given $c=10$, $a=12$, $B=35°$, find the length of the median which bisects BC.

12. Find the obtuse angle in the triangle whose sides are as 2 : 5 : 6.

13. Prove with help of figures (1) when A, B are acute, (2) when A is obtuse, that $c=a\cos B+b\cos A$.

14. In a triangle $A=115°$, $a=3$, $c=2$; find the other angles.

15. OX, OY are two straight roads inclined at 60°. A man A walks along OX at 4 miles an hour, and B starts along OY at the same time. If B is $7\frac{1}{2}$ miles from A at the end of 2 hours, obtain a quadratic equation for the distance B has walked in that time and solve it.

16. a, b, c, d are the sides of a quadrilateral inscribed in a circle, and θ is the angle contained by a, b; by writing down two expressions for the diagonal opposite θ, prove that

$$\cos\theta=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

17. If x be the length of a diagonal of a parallelogram which makes angles α, β with the sides, show that the sides are

$$\frac{x\sin\alpha}{\sin(\alpha+\beta)} \text{ and } \frac{x\sin\beta}{\sin(\alpha+\beta)}.$$

18. A straight line AD divides the angle A of a triangle ABC into two parts α, β and meets BC at D: show that

$$\frac{BD}{DC}=\frac{c\sin\alpha}{b\sin\beta}.$$

19. The parts a, c, A of a triangle are given. Write down a quadratic equation for the remaining side b. If b_1, b_2 are the lengths of the third side in the two triangles which have the given parts, show that $b_1+b_2=2c\cos A$ and $b_1b_2=c^2-a^2$.

Also prove that the sum of the areas of the two triangles is $c^2\sin A\cos A$, and consequently independent of a.

20. In the $\triangle$ABC, if the line joining A to the mid-point of BC is perpendicular to AC, prove

$$\cos A\cos C=\frac{2(c^2-a^2)}{3ac}.$$

Miscellaneous Examples. C.

1. Draw two straight lines OX, OY at right angles, OX to the right, OY up, and find a point P 4″ from OX and 1″ from OY. Now imagine P to remain fixed while YOX is revolved counter-clockwise about O. Determine both by drawing and calculation, (1) what amount of revolution would make OX pass through P; (2) the distance of P from OX when OX has turned through 60°.

2. Being given $\cos 41° 24' = \frac{3}{4}$, find two values of θ less than 180° which satisfy $4 \cos 2\theta + 3 = 0$.

3. In a triangle ABC prove that

$$\tan A = \frac{a \sin B}{c - a \cos B}.$$

4. A man surveying a mine measures a length AB of 12 chains due E. with a dip of 8° to the horizon; then a length BC of 20 chains due E. with a dip of 5°. How much deeper vertically is C than A? A chain = 66 ft. Give the answer in feet.

5. Two lines OA, OB of length r_1, r_2 respectively make angles of θ_1 and θ_2 with a third line OX. Prove

$$AB^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1).$$

---

6. In a triangle $\sin^2 C = \sin^2 A + \sin^2 B$. Prove that the triangle is right-angled.

7. ABCD is a parallelogram:

$$AB = 2{\cdot}5'', \quad BC = 4'', \text{ and } \angle ABC = 65°.$$

Calculate the area of the parallelogram and the length of the diagonals.

8. A is 200 yards from B in the same horizontal plane. The angular elevation at A of a kite vertically above B is 55° 30′. How far must the kite descend before its angular elevation as seen from A is half that angle?

9. If D be the mid-point of the side BC of an equilateral triangle ABC, and O the point of intersection of the medians, prove by finding the lengths of AD and AO in terms of a side of the triangle that AO = 2 . OD.

10. A mast is secured by 3 equal stays connecting its highest point with 3 pegs on the ground at the corners of an equilateral triangle. If the length of each stay is 45 feet and the distance between 2 pegs is 30 ft., find the height of the mast.

---

11. Find from the definitions a formula which will give $\cos\theta$ when $\tan\theta$ is known.

Taking $\theta = 34° 43'$, find its tangent from the tables: then find its cosine from your formula and compare the result with that given in the tables.

12. A balloon is vertically over a point which lies in a direct line between two observers who are 2000 ft. apart and who note the angles of elevation of the balloon to be $35° 30'$ and $61° 20'$: find its height.

13. The half ABC of a rectangular sheet of paper ABCD, AB $=5''$, BC $=7''$, is folded about the diagonal AC. Find by using tables the angle between CD and the new position of CB. Find also the length of the line joining the old and the new positions of B.

14. Two circles whose radii are 5 cms. and 3 cms. have their centres 10 cms. apart; prove that the common tangents make angles $\sin^{-1}{\cdot}2$, or $\sin^{-1}{\cdot}8$ with the line joining the centres.

15. A line of length x is drawn from A to any point in the side BC of a triangle ABC and makes angles of θ, ϕ respectively with AB, AC: prove by using the formula for the area of a triangle that

$$\frac{\sin\theta}{b} + \frac{\sin\phi}{c} = \frac{\sin(\theta+\phi)}{x}.$$

---

16. Take a line OA, length 5 cms., near the lower edge of the page and draw perpendicular to it a line AB of unlimited length. Find with your instruments eight points P, Q, R ... on AB such that AOP $=$ POQ $=$ QOR $= \ldots = 10°$. From the figure find the average increase of the tangent of the angle for each degree between 0° and 10°, 10° and 20°, etc.

What happens to the tangent as the angle increases from 80° to 90°?

17. Two adjacent sides of a parallelogram inclined at an angle a are P and Q. The diagonal passing through their point of intersection is R. Prove

$$R^2 = P^2 + Q^2 + 2PQ\cos a.$$

18. Find the positive values of θ between 0° and 360° which satisfy the equation

$$6\cot\theta + 1 = 12\tan\theta.$$

19. OX, OY are two straight lines intersecting at an angle θ. A point A is taken on OY such that $OA = a$, and then AB is drawn perpendicular to OY meeting OX in B; BC is drawn perpendicular to OX meeting OY in C and CD is drawn perpendicular to OY meeting OX in D. Prove that

$$CD = a\tan\theta\,(1 + \tan^2\theta).$$

20. From a point O three straight lines OA, OB, OC are drawn in the same plane of lengths 1, 2, 3 respectively and with the angles AOB, BOC each equal to 60°. Find the angle ABC.

CHAPTER VI.

PROJECTION. FORMULAE FOR COMPOUND ANGLES.

36. Projection.

If from the extremities of a line OP, perpendiculars OM, PN be dropped to another line AB, then MN is said to be the projection of OP on the line AB.

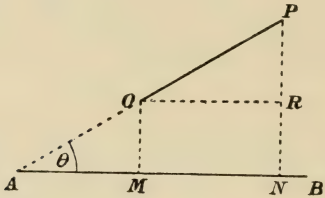

Fig. 49.

Length of projection.

Let the angle between OP and AB be θ.

Then from the diagram

$$MN = OR = OP \cos\theta,$$

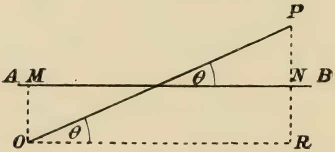

Fig. 50.

i.e. the length of the projection of a line OP on another line equals OP × cosine of angle OP makes with the line on which it is projected.

If $\theta = 90°$, then the projection of OP

$$= OP \cos 90° = 0.$$

If $\theta = 0°$ then the projection of OP

$$= OP \cos 0° = OP.$$

Exercise. (i) Take two fixed points P and Q and any straight line AB. Let R be any other point. Project PQ, PR, RQ on AB. Show by a diagram that by adopting the sign convention we have, for all positions of R :

Projection of PR + Projection of RQ = Projection of PQ.

(ii) Show by a diagram that the sum of the projections on any straight line, of the sides, taken in order, of any closed polygon is zero.

37. Trigonometrical ratios of compound angles.

It is frequently useful to express the trigonometrical ratios of compound angles such as $A + B$, or $A - B$, in terms of the ratios of A and B.

The beginner is apt to think that $\sin(A + B)$ is

$$= \sin A + \sin B,$$

a statement which can at once be shown to be incorrect by the help of tables.

For instance, $74^\circ = 40^\circ + 34^\circ$;

but $\sin 74^\circ = \cdot 9613$,

and $\sin 40^\circ + \sin 34^\circ = \cdot 6428 + \cdot 5592 = 1\cdot 2020$.

In the following articles we shall prove that

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B,$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Exercise.

Which is the greater, $\cos(A + B)$ or $\cos A$?

Why is the statement $\cos(A + B) = \cos A + \cos B$ obviously absurd?

38. To prove $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

Let XOQ be the angle A, and POQ the angle B.

In the line OP bounding the compound angle A + B take a point P and let fall a perpendicular PQ, on OQ.

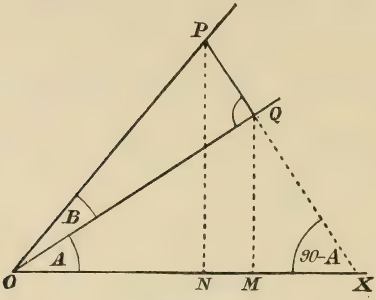

Fig. 51.

Project OQ, OP on OX.

Now $OM = ON + NM$,

i.e. projection of OQ = projection of OP + projection of PQ,

or $$OQ \cos A = OP \cos(A+B) + PQ \cos(90° - A)$$

(by producing PQ we see that the angle PQ makes with OX is $(90° - A)$ since OQP is a right angle),

or $$\frac{OQ}{OP} \cos A = \cos(A+B) + \frac{PQ}{OP} \sin A,$$

i.e. $$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

Note. If in this formula we write $-B$ for B, we get

$$\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$$
$$= \cos A \cos B + \sin A \sin B,$$

since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$.

39. To prove $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

With the same construction as before, but projecting on OY at right angles to OX,

$$ON = OM + MN,$$

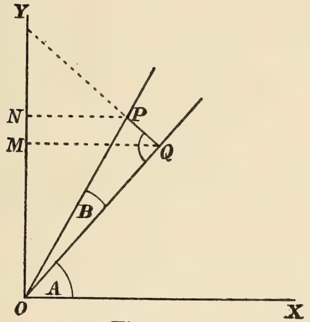

Fig. 52.

i.e. projection of OP = projection of OQ + projection of QP,

$$OP \cos[90° - (A+B)] = OQ \cos(90° - A) + QP \cos A.$$

By producing QP we see that the angle QP makes with OY is A, since it is the complement of QOY; ∴ we have

$$OP \sin(A+B) = OQ \sin A + QP \cos A,$$

or
$$\sin(A+B) = \frac{OQ}{OP} \sin A + \frac{QP}{OP} \cos A$$

$$= \sin A \cos B + \cos A \sin B.$$

Note. If for B we write $-B$, we get

$$\sin(A-B) = \sin A \cos B - \cos A \sin B.$$

40. Independent proofs that

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Let OP describe the angle A in the positive direction, and then the angle B in a negative direction.

As before, take a point P in the line OP bounding the compound angle $A - B$ and drop a perpendicular on OQ.

Project on OX for $\cos(A - B)$, on OY for $\sin(A - B)$,

$$ON = OM + MN;$$

projection of OP = projection of OQ + projection of QP,

$$OP\cos(A - B) = OQ\cos A + QP\cos(90^\circ - A),$$

from which

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Taking projections on OY, we have

$$OM' = ON' + N'M',$$

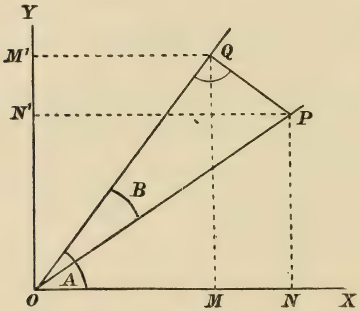

Fig. 53.

i.e. projection of OQ = projection of OP + projection of PQ;

$$\therefore\ OQ\cos(90^\circ - A) = OP\cos(90^\circ - \overline{A - B}) + PQ\cos A;$$

$$\therefore\ OQ\sin A = OP\sin(A - B) + PQ\cos A;$$

whence $\quad \sin(A - B) = \sin A \cos B - \cos A \sin B.$

Note. We have seen that the formula for $\cos(A-B)$ may be deduced from that for $\cos(A+B)$.

If we write $(90-A)$ for A in the formula for $\cos(A+B)$, we get

$$\cos(90-A+B)=\cos(90-A)\cos B-\sin(90-A)\sin B,$$

i.e. $\cos[90-(A-B)]$

$$=\sin(A-B)=\sin A\cos B-\cos A\sin B.$$

Similarly we can obtain the formula for

$$\sin(A+B).$$

Example (i).

Prove $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B.$

$$\sin(A+B)\sin(A-B)=(\sin A\cos B+\cos A\sin B)$$
$$(\sin A\cos B-\cos A\sin B$$
$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$$
$$=\sin^2 A(1-\sin^2 B)-(1-\sin^2 A)\sin^2 B$$
$$=\sin^2 A-\sin^2 B.$$

Example (ii).

Expand $\sin(A+B+C).$

Treating $(B+C)$ as a single angle we have

$$\sin[A+(B+C)]=\sin A\cos(B+C)+\cos A\sin(B+C)$$
$$=\sin A(\cos B\cos C-\sin B\sin C)$$
$$+\cos A(\sin B\cos C+\cos B\sin C)$$
$$=\sin A\cos B\cos C+\sin B\cos A\cos C$$
$$+\sin C\cos A\cos B-\sin A\sin B\sin C.$$

Example (iii).

Find the value of

$$\cos 34^\circ\cos 42^\circ-\sin 34^\circ\sin 42^\circ.$$

By comparing with the formula for $\cos(A+B)$ we see that this expression equals $\cos(34^\circ+42^\circ)=\cos 76^\circ=\cdot 2419$ (from the Tables).

Example (iv).

The following example shows the use made of Projection in Statical problems.

A weighted rod AB, 4 ft. long, is suspended by a string, fastened to its two ends, which passes over a pulley at O so that each portion is inclined at an angle of 35° to the vertical. The rod makes an angle of 20° with the horizon. Find the length of the string.

Let x and y be the lengths of the two portions of the string.

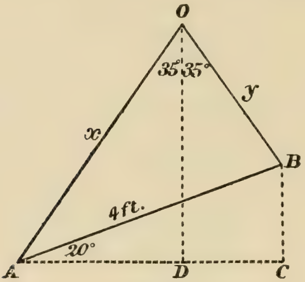

Fig. 54.

Project on AC the horizontal line through A.

Projection of AO + projection of OB = projection of AB;

$$\therefore \quad x \sin 35° + y \sin 35° = 4 \cos 20°;$$

$$\therefore \quad x + y = \frac{4 \cos 20°}{\sin 35°} = 6{\cdot}5;$$

$\therefore$ length of string = 6·5 ft. approximately.

Examples. VI a.

1. If $\cos\alpha=\frac{3}{5}$, and $\cos\beta=\frac{4}{5}$, calculate the values of

$\sin\alpha$, $\sin\beta$, $\sin(\alpha+\beta)$, $\cos(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\cos(\alpha-\beta)$.

2. If $\sin\alpha=\frac{1}{3}$, and $\sin\beta=\frac{1}{4}$, calculate the values of $\cos\alpha$, $\cos\beta$, $\sin(\alpha+\beta)$. Verify by finding the angles α, β by the help of the tables.

3. If $\cos\alpha=\cdot 2$ and $\cos\beta=\cdot 5$, find $\cos(\alpha-\beta)$ and verify from the tables.

4. Expand $\cos(90°-A)$, and show that it equals $\sin A$.

Expand also $\cos(180°+A)$, and $\sin(90°+A)$.

5. Find the values of

(i) $\sin 47° \cos 16° - \cos 47° \sin 16°$,

(ii) $\sin 52° \sin 27° - \cos 52° \cos 27°$.

6. By writing $\cos 75°$ as $\cos(45°+30°)$ and expanding, prove

$$\cos 75° = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

7. Prove that $\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}$, and find $\sin 15°$.

8. Prove that $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B$.

9. Show that $\sqrt{2}\sin(A+45°)=\sin A+\cos A$.

10. Prove that $\dfrac{\sin(A+B)}{\cos A\cos B}=\tan A+\tan B$.

11. Show that $\cos A-\sin A=\sqrt{2}\cos(A+45°)$.

12. Find the values of

(i) $\cos 18° \cos 36° - \sin 18° \sin 36°$,

(ii) $\sin 18° \cos 36° + \cos 18° \sin 36°$.

13. Prove that

$\sin(A+B)+\cos(A-B)=(\sin A+\cos A)(\sin B+\cos B)$.

14. Factorise $\sin(A-B)+\cos(A+B)$.

15. Expand $\cos(A+B+C)$.

16. If θ and ϕ are both less than 180°, and $\frac{\sin\theta\sin\phi}{\cos\theta\cos\phi} = -1$, show that θ and ϕ differ by a right angle.

17. A sphere of radius r rolls down an inclined plane which makes an angle a with the horizon. Prove that the height of the centre of the sphere above the horizontal plane when the point of contact of the sphere is at a distance l from the foot of the inclined plane is $r\cos a + l\sin a$.

18. OX, OY are two straight lines at right angles. P is a point 4″ from OX and 3″ from OY. Through O a straight line is drawn making an angle θ with OX. Prove by projection that the length of the perpendicular from P on this line is $4\cos\theta - 3\sin\theta$.

41. To prove

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

$$= \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

(dividing numerator and denominator by $\cos A\cos B$)

$$= \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

Prove in a similar way

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

Examples. VI b.

1. Prove that $\tan(45^\circ + A) = \dfrac{1 + \tan A}{1 - \tan A}$.

2. Find the value of $\dfrac{\tan 47^\circ - \tan 20^\circ}{\tan 20^\circ \tan 47^\circ + 1}$.

3. Prove that $\tan 75^\circ = 2 + \sqrt{3}$, and find $\tan 15^\circ$.

4. Expand $\tan(90^\circ + A)$ and show that it equals $-\cot A$.

5. In a similar way prove that

$$\tan(180^\circ + A) = \tan A,$$

and

$$\tan(180^\circ - A) = -\tan A.$$

6. By writing $\cot(A+B)$ as $\dfrac{\cos(A+B)}{\sin(A+B)}$ and expanding, prove that it equals $\dfrac{\cot A \cot B - 1}{\cot A + \cot B}$.

7. Express $\cot(A - B)$ in terms of $\cot A$ and $\cot B$.

8. Given $\tan \alpha = 1$ and $\tan(\alpha + \beta) = 2$, find $\tan \beta$.

9. If $\tan A = \frac{4}{5}$ and $\tan B = \frac{1}{9}$, show that $A + B = 45^\circ$, supposing A and B to be acute angles.

10. The perpendicular from the vertex of a triangle is 6″ long and it divides the base into segments which are 2″ and 3″ respectively. Find the tangent of the vertical angle.

11. ABC is an isosceles triangle, right angled at C, and D is the middle point of AC. Prove that DB divides the angle B into two parts whose cotangents are in the ratio 2 : 3.

12. If two straight lines make with a third straight line OX angles θ and θ', measured from the same direction OX, such that $\tan \theta = m$ and $\tan \theta' = m'$, prove that the angle between the two lines is

$$\tan^{-1} \frac{m \sim m'}{1 + mm'}.$$

13. Expand $\tan(\alpha + \beta + \gamma)$ first in terms of $\tan \alpha$ and $\tan(\beta + \gamma)$ and hence in terms of $\tan \alpha$, $\tan \beta$, $\tan \gamma$. Use your result to show that (i) if $\alpha + \beta + \gamma = 180^\circ$, then

$$\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma,$$

(ii) if $\alpha + \beta + \gamma = 90^\circ$, then

$$\tan \beta \tan \gamma + \tan \gamma \tan \alpha + \tan \alpha \tan \beta = 1.$$

14. A vertical pole more than 100 ft. high consists of two parts, the lower being $\frac{1}{3}$ of the whole. At a point in the horizontal plane through the foot of the pole and 40 ft. from it, the upper part subtends an angle whose tangent is $\frac{1}{2}$. Find the height of the pole.

42. To express $\sin 2A$, $\cos 2A$ and $\tan 2A$ as functions of A.

We have

$$\sin 2A = \sin(A + A) = \sin A \cos A + \cos A \sin A;$$

$$\therefore \ \sin 2A = 2 \sin A \cos A \dots\dots\dots\dots\dots(1).$$

Also

$$\cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A;$$

$$\therefore \ \cos 2A = \cos^2 A - \sin^2 A \dots\dots\dots\dots(2).$$

Writing $1 - \cos^2 A$ for $\sin^2 A$, we get

$$\cos 2A = 2\cos^2 A - 1 \dots\dots\dots\dots\dots(3).$$

Writing $1 - \sin^2 A$ for $\cos^2 A$, we get

$$\cos 2A = 1 - 2\sin^2 A \dots\dots\dots\dots\dots(4).$$

The results (3) and (4) may be written

$$1 + \cos 2A = 2\cos^2 A \dots\dots\dots\dots\dots(5),$$

$$1 - \cos 2A = 2\sin^2 A \dots\dots\dots\dots\dots(6),$$

and in this form are of much importance.

From (5) and (6) we have

$$\frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A.$$

Again,

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A};$$

$$\therefore \ \tan 2A = \frac{2\tan A}{1 - \tan^2 A} \quad \dots\dots\dots\dots(7).$$

It is important to notice that the above formulae enable us to express functions of an angle in terms of the functions of half the angle.

Thus
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2};$$
$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$
$$= 2\cos^2\frac{\theta}{2} - 1$$
$$= 1 - 2\sin^2\frac{\theta}{2};$$
$$\sin 3\theta = 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2};$$
$$\tan\frac{\theta}{2} = \frac{2\tan\frac{\theta}{4}}{1-\tan^2\frac{\theta}{4}}.$$

Note. The expression $1-\cos\theta$ is of frequent occurrence in Nautical computations and is called versine θ. Half-versine is contracted to Haversine and from the formula $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, we see, $\text{hav}\,\theta = \frac{\text{vers}\,\theta}{2} = \frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$.

Example (i).

Prove
$$\cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}.$$

We have
$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$$
$$= \cos^2\frac{A}{2}\left(1-\tan^2\frac{A}{2}\right)$$
$$= \frac{1-\tan^2\frac{A}{2}}{\sec^2\frac{A}{2}}$$
$$= \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}.$$

Example (ii).

Prove $\sin 3A = 3\sin A - 4\sin^3 A$.

We have

$$\begin{aligned}\sin 3A &= \sin(2A + A)\\ &= \sin 2A\cos A + \cos 2A\sin A\\ &= 2\sin A\cos^2 A + (1 - 2\sin^2 A)\sin A\\ &= 2\sin A(1 - \sin^2 A) + (1 - 2\sin^2 A)\sin A\\ &= 3\sin A - 4\sin^3 A.\end{aligned}$$

Exercise.

Prove in a similar way that

$$(1)\ \sin A = \frac{2\tan\frac{A}{2}}{1 + \tan^2\frac{A}{2}};\quad (2)\ \cos 3A = 4\cos^3 A - 3\cos A;$$

$$(3)\ \tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Examples. VI c.

1. If $\sin a = \frac{4}{5}$, calculate $\cos a$, $\sin 2a$, $\cos 2a$.
2. Given $\cos a = \cdot 4$, find $\sin 2a$, $\cos 2a$, $\tan 2a$.
3. Find the value of $2\sin 25^\circ \cos 25^\circ$, $1 - 2\sin^2 25$.
4. Prove that $(\sin\theta - \cos\theta)^2 = 1 - \sin 2\theta$.
5. Find $\tan 2A$ when $\tan A = \cdot 5$.
6. Factorise $\cos^4 A - \sin^4 A$ and prove it equal to $\cos 2A$.
7. If $\cos 2a = \frac{2}{3}$, prove $\tan a = \frac{\sqrt{5}}{5}$.
8. If $1 + \cos 2a = \frac{32}{25}$, find $\cos a$.
9. If $1 - \cos 2a = \frac{9}{8}$, find $\sin a$.
10. Given that $\tan a = \frac{1}{3}$, prove $\cos 2a = \frac{4}{5}$.

11. Find the values of $\sqrt{\dfrac{1-\cos 56^\circ}{2}}$ and $\sqrt{\dfrac{1+\cos 56^\circ}{2}}$.

12. Find the value of $\sqrt{\dfrac{1-\cos 40^\circ}{1+\cos 40^\circ}}$.

13. Find $\tan\dfrac{a}{2}$, given $\cos a = \frac{3}{5}$.

14. Find the value of $2\cos 2\theta + 3\sin 2\theta$ when $\tan\theta = \frac{3}{2}$.

15. Prove that $1-\cos a\cos\beta - \sin a\sin\beta = 2\sin^2\dfrac{a-\beta}{2}$.

16. Find the positive values of A between 0° and 360° which satisfy the equations

(i) $\cos 2A + \sin^2 A = \frac{3}{4}$; (ii) $\tan 2A = 3\tan A$.

17. Express $\cos 4a$ in terms of $\cos a$.

18. Find the value of $a\cos 2\theta + b\sin 2\theta$ when $\tan\theta = \dfrac{b}{a}$.

19. If $\cot\theta = \dfrac{1-\cos\phi}{\sin\phi}$, prove that $\dfrac{\phi}{2} = 90^\circ - \theta$.

20. Express $\cos^2 a - \sin^2\beta$ as half the sum of two cosines and hence evaluate $\cos^2 63^\circ - \sin^2 47^\circ$.

21. If $\tan\theta = \dfrac{b}{a}$, simplify $\tan 2\theta + \sec 2\theta$.

22. If $\cot^2\theta - \cot\theta = 1$, prove $\cot 2\theta = \frac{1}{2}$.

23. AB is the diameter of a circle of radius r, whose centre is at C. P is a point on the circumference where $\angle BCP = \theta$. Prove that the projection of AP on the diameter equals $2r\cos^2\dfrac{\theta}{2}$. Shew that this result is true whether θ is acute or obtuse.

24. A point P moves round the circumference of a wheel of radius r, centre O, placed in a vertical plane. If A is the lowest position of P show that the vertical height of P above A at any time is $2r\sin^2\dfrac{\theta}{2}$ where $\angle AOP = \theta$.

25. Two radii OP, OQ of a circle of radius r are inclined at an angle θ. The perpendicular from O on PQ cuts the chord at A and the arc at B. Prove $AB = 2r\sin^2\dfrac{\theta}{4}$.

43. The formulae of Article 37 are useful for obtaining solutions of equations of the form

$$a \sin\theta + b\cos\theta = c.$$

Example.

Find a solution of the equation $3\sin\theta - 2\cos\theta = 2$.

Let a be an acute angle such that $\tan a = \frac{2}{3}$; then

$$\sin a = \frac{2}{\sqrt{13}}, \quad \cos a = \frac{3}{\sqrt{13}}.$$

The equation can now be written

$$\sqrt{13}\,(\sin\theta\cos a - \cos\theta\sin a) = 2\,;$$

whence
$$\sin(\theta - a) = \frac{2}{\sqrt{13}}$$
$$= \frac{2\sqrt{13}}{13}$$
$$= \frac{2 \times 3{\cdot}606}{13}$$
$$= {\cdot}5548$$
$$= \sin 33^\circ\ 42'.$$

Also $\qquad a = \tan^{-1}\frac{2}{3} = \tan^{-1}{\cdot}6667 = 33^\circ\ 41'$;

$\therefore$ a solution of the equation is given by

$$\theta - 33^\circ\ 41' = 33^\circ\ 42'\,;$$

whence
$$\theta = 67^\circ\ 23'.$$

The angle a which has been introduced in the work is called a **subsidiary angle.** Other occasions when a subsidiary angle is of use will be found in Articles 56, 57.

Beginners sometimes solve equations of the form

$$a\cos\theta + b\sin\theta = c$$

by substituting $\sqrt{1-\sin^2\theta}$ for $\cos\theta$ and squaring: but this method is not satisfactory, as in consequence of squaring we obtain some values of θ which are not roots of the given equation.

Examples. VI d.

1. Show that $3\sin\theta + 4\cos\theta = 5\sin(\theta + a)$, where $a = \tan^{-1}\frac{4}{3}$; and hence prove that the greatest value of $3\sin\theta + 4\cos\theta$, when θ may have any value, is 5. What is the value of θ in this case?

2. Find a solution of $3\sin\theta + 4\cos\theta = 2$.

3. Find a value of x which satisfies $\cos x + \sin x = \cdot 5$.

4. Find a solution of $4\cos x - 3\sin x = 3$.

5. If $\frac{p}{q} = \tan\theta$, prove that

$$p\cos a - q\sin a = \sqrt{p^2 + q^2}\sin(\theta - a),$$

and find the greatest value of $p\cos a - q\sin a$ if a varies.

44. We have proved

(i) $\sin A \cos B + \cos A \sin B = \sin(A + B)$,

(ii) $\sin A \cos B - \cos A \sin B = \sin(A - B)$,

(iii) $\cos A \cos B - \sin A \sin B = \cos(A + B)$,

(iv) $\cos A \cos B + \sin A \sin B = \cos(A - B)$.

Adding (i) and (ii), we get

(α) $$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$
$$= \sin(\text{sum}) + \sin(\text{difference}).$$

Subtracting (i) and (ii)

(β) $$2\cos A \sin B = \sin(A + B) - \sin(A - B)$$
$$= \sin(\text{sum}) - \sin(\text{difference}).$$

Adding (iii) and (iv)

(γ) $$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$
$$= \cos(\text{sum}) + \cos(\text{difference}).$$

Subtracting (iii) from (iv)

$\{$since $(A + B) > (A - B)$; $\therefore \cos(A + B) < \cos(A - B)\}$.

(δ) $$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$
$$= \cos(\text{difference}) - \cos(\text{sum}).$$

These formulae enable us to express products of sines and cosines as sums or differences, and should be learnt in the verbal form.

It will be noticed that in both (α) and (β) we have the product of a sine and a cosine; but either formula gives the same result, as will be seen from the following example.

Example (i).

$$\begin{aligned} 2\sin 5\theta \cos 2\theta &= \sin(\text{sum}) + \sin(\text{difference}) \\ &= \sin(5\theta + 2\theta) + \sin(5\theta - 2\theta) \\ &= \sin 7\theta + \sin 3\theta. \end{aligned}$$

If however we apply formula (β) which also gives the product of a sine and a cosine, we have

$$\begin{aligned} 2\cos 2\theta \sin 5\theta &= \sin(\text{sum}) - \sin(\text{difference}) \\ &= \sin(2\theta + 5\theta) - \sin(2\theta - 5\theta) \\ &= \sin 7\theta - \sin(-3\theta) \\ &= \sin 7\theta + \sin 3\theta, \end{aligned}$$

for $\qquad \sin(-3\theta) = -\sin 3\theta.$

Example (ii).

$$\begin{aligned} \cos 2\theta \cos 5\theta &= \tfrac{1}{2}[\cos(\text{sum}) + \cos(\text{difference})] \\ &= \tfrac{1}{2}[\cos(2\theta + 5\theta) + \cos(2\theta - 5\theta)] \\ &= \tfrac{1}{2}[\cos 7\theta + \cos(-3\theta)] \\ &= \frac{\cos 7\theta + \cos 3\theta}{2}, \end{aligned}$$

since $\qquad \cos(-3\theta) = \cos 3\theta.$

Example (iii).

$$\begin{aligned} 2\sin 5\theta \sin 2\theta &= \cos(\text{difference}) - \cos(\text{sum}) \\ &= \cos(5\theta - 2\theta) - \cos(5\theta + 2\theta) \\ &= \cos 3\theta - \cos 7\theta. \end{aligned}$$

Examples. VI e.

Express as the sum or difference of two Trigonometrical ratios: verify approximately the numerical examples by help of the tables.

1. $2\sin 3\theta\cos\theta$.
2. $2\cos 3\theta\cos\theta$.
3. $\sin 3\theta\sin\theta$.
4. $2\cos 3\theta\sin\theta$.
5. $\sin A\cos 2A$.
6. $\sin A\cos B$.
7. $\cos 2A\cos 2B$.
8. $\sin 5\theta\sin\theta$.
9. $2\sin 20^\circ\cos 70^\circ$.
10. $2\cos 40^\circ\cos 30^\circ$.
11. $2\sin 10^\circ\sin 20^\circ$.
12. $\cos 50^\circ\cos 30^\circ$.
13. $2\sin(A+B)\cos(A-B)$.
14. $2\cos(A+2B)\cos(2A+B)$.
15. $2\cos\frac{A}{2}\sin\frac{A}{2}$.
16. $\sin 3a\sin a$.

45. The formulae of Article 44 give us sums and differences expressed as products, but it is more convenient to put the formulae in a different form, as follows.

Writing X for $(A+B)$, and Y for $(A-B)$, we have

$$A+B=X,$$
$$A-B=Y;$$
$$\therefore\ 2A=X+Y,\ \text{ or }\ A=\frac{X+Y}{2};$$

and
$$2B=X-Y,\ \text{ or }\ B=\frac{X-Y}{2}.$$

Substituting in (α), (β), (γ), (δ), of Article 44, we get

from (α) $$\sin X+\sin Y=2\sin\frac{X+Y}{2}\cos\frac{X-Y}{2},$$

i.e. **sum of sines = 2 sin (half sum) cos (half difference)**;

from (β) $$\sin X-\sin Y=2\cos\frac{X+Y}{2}\sin\frac{(X-Y)}{2},$$

i.e. **difference of sines = 2 cos (half sum) sin (half difference)**;

from (γ) $$\cos X+\cos Y=2\cos\frac{X+Y}{2}\cos\frac{X-Y}{2},$$

sum of cosines = 2 cos (half sum) cos (half difference);

from (δ) $$\cos Y-\cos X=2\sin\frac{X+Y}{2}\sin\frac{X-Y}{2},$$

difference of cosines = 2 sin (half sum) sin (half difference <u>reversed</u>).

Example (i).

Express as a product $\sin 3\theta + \sin 2\theta$,

$$\sin 3\theta + \sin 2\theta = 2 \sin (\text{half sum}) \cos (\text{half difference})$$

$$= 2 \sin \frac{5\theta}{2} \cos \frac{\theta}{2}.$$

Example (ii).

$$\cos 3\theta - \cos 5\theta = 2 \sin (\text{half sum}) \sin (\text{half difference reversed})$$

$$= 2 \sin \frac{3\theta + 5\theta}{2} \cdot \sin \frac{5\theta - 3\theta}{2}$$

$$= 2 \sin 4\theta \sin \theta.$$

Example (iii).

Prove that

$$\sin a - \sin 2a + \sin 3a = 4 \sin \frac{a}{2} \cos a \cos \frac{3a}{2},$$

$$\sin a - \sin 2a + \sin 3a = \sin a + \sin 3a - \sin 2a$$

$$= 2 \sin 2a \cos a - 2 \sin a \cos a$$

$$= 2 \cos a (\sin 2a - \sin a)$$

$$= 2 \cos a \, 2 \cos \frac{3a}{2} \cdot \sin \frac{a}{2}$$

$$= 4 \sin \frac{a}{2} \cos a \cos \frac{3a}{2}.$$

Examples. VI f.

Express as products :

1. $\sin 3A + \sin A$.
2. $\sin 3A - \sin A$.
3. $\cos 3A + \cos A$.
4. $\cos A - \cos 3A$.
5. $\sin 2\theta - \sin \theta$.
6. $\cos 3\theta - \cos 2\theta$.
7. $\sin B + \sin A$.
8. $\cos 2a + \cos 2\beta$.
9. $\cos 2a - \cos 2\beta$.
10. $\sin 23° + \sin 14°$.
11. $\cos 32° - \cos 41°$.
12. $\sin 41° + \cos 12°$.
13. $\cos 18° + \cos 43°$.
14. Prove that
$$\frac{\sin \theta + \sin \phi}{\sin \theta - \sin \phi} = \tan \frac{\theta + \phi}{2} \cot \frac{\theta - \phi}{2}.$$
15. Prove $\dfrac{\cos \theta - \cos \phi}{\cos \theta + \cos \phi} = -\tan \dfrac{\theta + \phi}{2} \tan \dfrac{\theta - \phi}{2}.$

16. Prove $$\frac{\sin 5^\circ + \sin 47^\circ}{\cos 5^\circ - \cos 47^\circ} = \tan 69^\circ.$$

17. Prove $$\frac{\sin 10^\circ + \sin 26^\circ}{\cos 10^\circ + \cos 26^\circ} = \cot 72^\circ.$$

18. If

$$x(\sin\theta - \sin\phi) + y(\cos\phi - \cos\theta) + \cos\theta\sin\phi - \sin\theta\cos\phi = 0,$$

show that $$x\cos\frac{\theta+\phi}{2} + y\sin\frac{\theta+\phi}{2} = \cos\frac{\theta-\phi}{2}.$$

19. Prove $\cos 3a \sin 2a - \cos 4a \sin a = \cos 2a \sin a.$

20. If $$x\cos a + y\sin a - c = 0,$$

and $$x\cos\beta + y\sin\beta - c = 0,$$

prove that $$x = \frac{c\cos\frac{a+\beta}{2}}{\cos\frac{a-\beta}{2}}, \quad y = \frac{c\sin\frac{a+\beta}{2}}{\cos\frac{a-\beta}{2}}.$$

21. In any triangle prove that

$$\frac{a+b}{c} = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}}.$$

22. If $x\cos\beta + y\cos a = p$ and $x\sin\beta - y\sin a = 0$,

prove $$x = \frac{p\sin a}{\sin(a+\beta)} \text{ and } y = \frac{p\sin\beta}{\sin(a+\beta)}.$$

23. If

$$\cos\theta = \frac{\cos u - e}{1 - e\cos u}, \text{ prove } \tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \cdot \tan\frac{u}{2}.$$

24. From the equations

$$T_1\cos\theta + T_2\cos\phi = 100,$$
$$T_1\sin\theta - T_2\sin\phi = 0,$$

show that $$T_1 = \frac{100\sin\phi}{\sin(\theta+\phi)} \text{ and } T_2 = \frac{100\sin\theta}{\sin(\theta+\phi)}.$$

25. Prove $$\frac{\cos 40^\circ + \cos 12^\circ}{\cos 40^\circ - \cos 12^\circ} = \tan 116^\circ . \cot 14^\circ.$$

Miscellaneous Examples. D.

1. Two straight lines make with another line angles, measured in the same direction, whose tangents are m and m'. If these two lines are at right angles, prove $1+mm'=0$. What is the relation between m and m' if the lines are parallel?

2. If α, β are the angles which satisfy the equation $4\tan^2\theta-3\tan\theta-2=0$, find the value of $\tan\alpha+\tan\beta$, $\tan\alpha\tan\beta$, $\tan(\alpha+\beta)$.

3. AB is a diameter of a circle of radius 5·6 ft. At A a line AC is drawn meeting the circle at C and the tangent at B in D. If $\mathrm{B\hat{A}C}=32^\circ\ 45'$, find the length of CD. Also if O be the centre and OD cuts the circumference in E, find the length of DE.

4. One mast of a ship is 12 feet longer than the other and both slope towards the stern at an angle of 10° to the vertical. The line joining their tops is inclined at 40° to the horizon. Find the horizontal distance between the masts.

5. If $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$ and $\frac{x}{a}\sin\phi-\frac{y}{b}\cos\phi=-1$,

prove
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

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6. The angles α and β are acute, $\sin\alpha=\frac{4}{5}$ and $\sin\beta=\frac{5}{13}$. Calculate the value of $\sin(\alpha+\beta)$ and of $\alpha+\beta$.

Construct a Δ ABC in which AD the perpendicular from A on BC is 6 cms. long and the angles DAB, DAC are the angles α, β. Measure the angle BAC and compare it with the value already found for $\alpha+\beta$.

7. Solve $a^2=b^2+c^2-2bc\cos\mathrm{A}$ as a quadratic equation in which b is the unknown quantity. And hence, or otherwise, calculate the positive value of b when $a=11$ cms., $c=9$ cms., $\mathrm{A}=40^\circ$. Check your result by an accurate drawing.

8. A hemispherical bowl, centre C, radius r, rests with its lowest point O on a horizontal plane. It is tilted until the line CO makes an angle θ with the vertical. Prove that the height of O above the plane is now $2r\sin^2\frac{\theta}{2}$.

9. A ladder 30 ft. long just reaches the top of a house and makes an angle of 67° with the ground. It is let down until it rests on a sill and then makes an angle of 48° with the ground. How far is the sill vertically below the point where the ladder first rested?

10. The angles which satisfy the equation

$$\tan^2\theta - 3\tan\theta - 1 = 0$$

are α and β. Prove that the difference between α and β is 90°.

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11. Solve the equations

$$x\sin\beta + y\cos\beta = 1,$$
$$x\cos\alpha + y\sin\alpha = 0.$$

12. The sides of a parallelogram are a, b, and the angle between them is θ. Prove that (1) the sum of the squares on the diagonals is $2(a^2+b^2)$; (2) the difference of the squares on the diagonals is $4ab\cos\theta$.

13. Three lines OA, OB, OC of length r_1, r_2, r_3 are drawn making angles θ_1, θ_2, θ_3 with the horizontal through O, prove that the area of the triangle ABC is

$$\tfrac{1}{2}[r_2r_3\sin(\theta_3-\theta_2)+r_3r_1\sin(\theta_1-\theta_3)+r_1r_2\sin(\theta_2-\theta_1)].$$

14. ABC is a triangle, $B=90°$, $BA=2$, $BC=3$, CD is the median joining C to the mid-point of AB. Prove that

$$\angle ACD = \tan^{-1}\tfrac{3}{11}.$$

15. The sights of a gun are 2 ft. apart and the back sight is raised till it is 2″ above the front sight when the barrel of the gun is pointing horizontally. I raise the gun till the line of sights points directly towards the top of a tower 100 ft. high and 500 yards distant. Find the tangent of the angle of elevation at which the barrel points and hence calculate the angle.

CHAPTER VII.

LOGARITHMS.

46. Definition.

The *logarithm* of a number to a given base is the index of the power to which the base must be raised in order to equal the number.

Thus if $x = e^y$, then y is the logarithm of x to the base e. This is written

$$\log_e x = y.$$

Example.

Find $\log_3 \sqrt{27}$.

Let $x = \log_3 \sqrt{27}$,

then $3^x = \sqrt{27}$

$= 3^{\frac{3}{2}}$;

$\therefore \; x = \frac{3}{2}$.

For practical purposes the base to which logarithms are calculated is 10; such logarithms are called common logarithms, and we shall confine ourselves to them.

Thus log 17 denotes the logarithm of 17 to the base 10.

47. In the first place we must prove certain fundamental laws of logarithms, on which the utility of logarithms depends.

I. $$\log ab = \log a + \log b.$$

Let $\log a = x$, and $\log b = y$.

Then $a = 10^x$, and $b = 10^y$;

$$\therefore\ ab = 10^x \times 10^y = 10^{x+y};$$

$\therefore$ by definition
$$\log ab = x + y$$
$$= \log a + \log b.$$

II. $$\log \frac{a}{b} = \log a - \log b.$$

We have
$$\frac{a}{b} = \frac{10^x}{10^y} = 10^{x-y};$$
$$\therefore\ \log \frac{a}{b} = x - y$$
$$= \log a - \log b.$$

III. $$\log a^n = n \log a.$$

We have
$$a^n = (10^x)^n = 10^{nx};$$
$$\therefore\ \log a^n = nx$$
$$= n \log a.$$

Examples.

$$\log(35 \times 4\cdot7) = \log 35 + \log 4\cdot7,$$
$$\log \tfrac{213}{421} = \log 213 - \log 421,$$
$$\log \sqrt{57} = \log 57^{\frac{1}{2}} = \tfrac{1}{2} \log 57,$$
$$\log \frac{34\sqrt{29}}{53} = \log 34 + \tfrac{1}{2} \log 29 - \log 53.$$

48. An inspection of the following table will enable us to formulate rules for writing down at sight the integral part of the logarithm of a number.

$$\begin{array}{lll}
10^4 = 10{,}000, & & \therefore \log 10{,}000 = 4; \\
10^3 = 1{,}000, & & \therefore \log 1{,}000 = 3; \\
10^2 = 100, & & \therefore \log 100 = 2; \\
10^1 = 10, & & \therefore \log 10 = 1; \\
10^0 = 1, & & \therefore \log 1 = 0; \\
10^{-1} = \frac{1}{10} & = \cdot 1, & \therefore \log \cdot 1 = -1; \\
10^{-2} = \frac{1}{100} & = \cdot 01, & \therefore \log \cdot 01 = -2; \\
10^{-3} = \frac{1}{1000} & = \cdot 001, & \therefore \log \cdot 001 = -3.
\end{array}$$

It will be noticed that the only numbers whose logarithms are whole numbers are those which are integral powers of 10.

The logarithms of numbers which lie between these various powers of 10 will be partly integral and partly decimal: thus, since 126·4 lies between 100 and 1000 its logarithm will lie between 2 and 3,

$$\text{i.e.} \quad \log 126{\cdot}4 = 2 + \text{a decimal}.$$

The integral part of the logarithm is called the *characteristic.*

The decimal part is called the *mantissa*, and it is always arranged that the mantissa is positive. The mantissa is obtained from Tables, as will shortly be explained; and the characteristic is found as follows.

All numbers with only one digit in the integral part have 0 as the characteristic of their logarithm; hence the characteristic for any number is the index of the power of ten by which the number must be divided in order that it may have one digit in the integral part, thus:

$$261{\cdot}3 = 2{\cdot}613 \times 10^2;$$

$$\begin{aligned}
\therefore \log 261{\cdot}3 &= \log 2{\cdot}613 + \log 10^2 \\
&= 0{\cdot}4171 + 2 \\
&\quad \text{(the mantissa being taken from the tables)} \\
&= 2{\cdot}4171.
\end{aligned}$$

Again $\cdot 002613 = 2{\cdot}613 \times 10^{-3}$;

$$\therefore \log \cdot 002613 = \log 2{\cdot}613 + \log 10^{-3}$$
$$= 0{\cdot}4171 - 3$$
$$= \bar{3}{\cdot}4171.$$

The negative sign is written over the 3 since the characteristic only is negative, the mantissa remaining positive. We write the logarithm in this form, and not $-2{\cdot}5829$, since by this device the mantissa will remain unaltered for all numbers having the same significant figures.

Various other mnemonics are often given for writing down characteristics, and are here stated for the benefit of those who prefer to use them.

1. The characteristic of the logarithm of a number which is greater than one is *one less* than the number of digits before the decimal point.

The characteristic of the logarithm of a number less than one is negative, and is *one more* than the number of zeros that follow the decimal point *or* is the same as the number of the place in which the first significant figure occurs.

2. Begin at the first significant figure and count the digits to the unit figure (not including the unit figure), this rule applying whether the number is greater or less than one.

Example.

Given $\log 2933 = 3{\cdot}4673$,

we have $\log 29{\cdot}33 = 1{\cdot}4673$;

for the characteristic is 1, since there are 2 digits in the integral part, and the mantissa remains unaltered.

Similarly $\log \cdot 002933 = \bar{3}{\cdot}4673$.

Again, we have $\cdot 4673 = \log 2{\cdot}933$; for there can only be one digit in the integral part, since the characteristic is zero, and $\cdot 4673$ is the mantissa corresponding to the digits 2933.

Similarly $\bar{2}{\cdot}4673 = \log \cdot 02933$.

Examples. VII a.

1. Write down the characteristics of the logarithms of the following numbers.

12·8, 161·4, ·3279, ·061, 1538,
2·749, ·0006, 13864, ·002, ·87.

2. Given that $\log 4023 = 3{\cdot}6045$, *write down*

$\log 4{\cdot}023$, $\log 402{\cdot}3$, $\log {\cdot}4023$, $\log {\cdot}004023$, $\log 40230$.

3. Given that $\log 2174 = 3{\cdot}3373$, *write down* the numbers whose logarithms are

$1{\cdot}3373$, $\bar{2}{\cdot}3373$, ${\cdot}3373$, $4{\cdot}3373$, $\bar{3}{\cdot}3373$, $2{\cdot}3373$, $\bar{1}{\cdot}3373$.

4. Given $\log 2 = {\cdot}3010$ and $\log 3 = {\cdot}4771$, find the logarithms of: 4, 5, 6, 8, 9, 12, 15, 16, 18, 20.

Also since approximately $7^4 = 2400$, $11^2 = 120$, $19^2 = 360$, find roughly $\log 7$, $\log 11$, $\log 14$, $\log 19$.

Taking difference of logs proportional to small difference in the numbers, find $\log 13$ since $\log 130$ lies between $\log 128$ and $\log 132$.

Now since $17 \times 10 = 169$ (approximately), find $\log 17$.

49. To obtain the logarithm of any number we write down the characteristic by rule, and obtain the mantissa from the tables as follows.

For purposes of explanation we give the following extract from Bottomley's Four Figure Tables:

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 2	3 4 5	5 6 7

From this portion of a page we read that the mantissa corresponding to 574 is ·7589 (note that the decimal point is not printed in the tables), and so we have

$$\log 574 = 2{\cdot}7589,$$
$$\log 57400 = 4{\cdot}7589,$$
$$\log {\cdot}0574 = \bar{2}{\cdot}7589.$$

If we require the mantissa corresponding to 4 digits, we must add on the difference obtained from the right hand of the page.

Thus mantissa for 574 is ·7589,
diff. for 6 is 5;
∴ mantissa for 5746 is ·7594.

After a little practice the student will have no difficulty in adding the difference mentally.

50. The reverse operation, to find the digits corresponding to a given mantissa, can be easily performed with the same tables; but more quickly with tables of antilogarithms, as shown below.

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	123	456	789
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1 3 4	5 7 8	9 10 12

Example.

Find x, being given $\log x = 2{\cdot}7594$.

From the extract of the tables given above, we have

·759 is the mantissa for 5741
4 is the difference for 5;
∴ ·7594 is the mantissa for 5746.

Since the characteristic is 2, we must have 3 digits in the integral part.

$$\therefore x = 574{\cdot}6.$$

Examples. VII b.

1. Write down the logarithms of

473, 4·735, ·2864, 456000, 87·67, ·003724.

2. Write down the numbers whose logarithms are

·4726, 3·7458, $\bar{1}{\cdot}8642$, 4·2175, $\bar{3}{\cdot}6847$.

51. Examples to illustrate the use of logarithms.

Example (i).

Find the value of

$$\frac{3{\cdot}562 \times {\cdot}06875}{({\cdot}7843)^2}.$$

If we denote the fraction by x, we have

$$\log x = \log 3{\cdot}562 + \log {\cdot}06875 - 2\log {\cdot}7843,$$

$$\begin{aligned} \log 3{\cdot}562 &= {\cdot}5516, \\ \log {\cdot}06875 &= \bar{2}{\cdot}8373 \\ &\quad \bar{1}{\cdot}3889 \text{ (by addition)}, \\ 2\log {\cdot}7843 = 2 \times \bar{1}{\cdot}8945 &= \bar{1}{\cdot}7890 \text{ (since } -2 + 1{\cdot}7890 = \bar{1}{\cdot}7890), \\ \log x &= \bar{1}{\cdot}5999 \text{ (by subtraction)}; \\ \therefore\ x &= {\cdot}398(0) \text{ (from antilogarithm tables)}. \end{aligned}$$

N.B. (i) For addition and subtraction arrange logarithms in columns.

(ii) The result is only correct to 3 significant figures but the fourth figure gives an approximation to the correct value which is ·3981 to four significant figures.

Example (ii).

Evaluate $\sqrt[3]{{\cdot}0276}$.

Let $x = \sqrt[3]{{\cdot}0276}$,

then

$$\begin{aligned} \log x &= \tfrac{1}{3}\log {\cdot}0276 \\ &= \tfrac{1}{3} \text{ of } \bar{2}{\cdot}4409 \\ &= \tfrac{1}{3} \text{ of } (-3 + 1{\cdot}4409) \text{ (see note)} \\ &= -1 + {\cdot}4803 \\ &= \bar{1}{\cdot}4803; \\ \therefore\ x &= {\cdot}3022. \end{aligned}$$

Note. Since the negative characteristic is not exactly divisible by the divisor 3, it is increased until it is a multiple of the divisor, proper correction being made.

Example (iii).

Find the reciprocal of 275·4.

Let $x = \dfrac{1}{275\cdot4} = (275\cdot4)^{-1}$.

Then $\log x = -\log 275\cdot4$

$= -2\cdot4399$ (both integral and decimal part being negative)

$= -3 + 1 - \cdot4399 = -3 + (1 - \cdot4399)$

$= \bar{3}\cdot5601$ (making the mantissa positive);

$\therefore\ x = \cdot003632$.

Example (iv).

Solve $575 \times (1\cdot03)^x = 847$.

We have, by taking logarithms of both sides,

$$\log 575 + x \log 1\cdot03 = \log 847;$$

$$\therefore\ x = \frac{\log 847 - \log 575}{\log 1\cdot03}$$

$$= \frac{\cdot1682}{\cdot0128}$$

$$= 13\cdot(1).$$

$$\begin{array}{r} 2\cdot9279 \\ \underline{2\cdot7597} \\ \cdot1682 \end{array}$$

$$128\,)\,1682\,(\,13$$
$$\begin{array}{r} 402 \\ 18 \end{array}$$

Note. We cannot obtain x to a greater degree of accuracy without using tables which give more than 4 figures.

The above equation gives the number of years in which £575 would amount to £847 at 3 % compound interest.

For the interest on £1 for 1 year = £·03;

$\therefore$ in 1 year £1 amounts to £1·03.

During the second year each £1 in this amounts to £1·03;

$\therefore$ £1·03 amounts to $\dfrac{1\cdot03}{1} \times$ £1·03 = £$(1\cdot03)^2$, and so on.

$\therefore$ after x years £1 amounts to £$(1\cdot03)^x$ and £575 to £$575 \times (1\cdot03)^x$.

52. Change of base.

If the logarithms of numbers to any base are known it is easy to obtain the logarithms to any other base.

Suppose logarithms to any base a are known and we wish to obtain the logarithm of any number n to the base b.

Let $$\log_b n = x,$$
then $$n = b^x;$$
$$\therefore \log_a n = \log_a b^x$$
$$= x \log_a b;$$
$$\therefore x = \frac{\log_a n}{\log_a b}.$$

Hence to transform logarithms calculated to base a to logarithms calculated to base b, we only have to multiply by $\dfrac{1}{\log_a b}$.

This multiplier is commonly called the modulus.

Examples. VII c.

Evaluate to three significant figures. State the fourth significant figure obtained although it cannot be relied upon as correct.

1. $23{\cdot}61 \times {\cdot}0324 \times 1{\cdot}384$.

2. $\dfrac{23{\cdot}68}{2{\cdot}174}$.

3. $\dfrac{{\cdot}0362}{{\cdot}004671}$.

4. $\dfrac{{\cdot}0264 \times 123{\cdot}6 \times 18{\cdot}41}{{\cdot}00326 \times 106{\cdot}4}$.

5. $\dfrac{21{\cdot}63 \times \sqrt{12{\cdot}18}}{361{\cdot}8}$.

6. $\dfrac{1}{23{\cdot}68}$.

7. $\dfrac{1}{{\cdot}0036 \times 2{\cdot}173}$.

8. $\dfrac{1{\cdot}274 \times {\cdot}0623 \times {\cdot}001}{2{\cdot}718 \times {\cdot}000526}$.

9. $\sqrt[3]{2174}$.

10. $(31{\cdot}76)^{\frac{3}{4}}$.

11. $\sqrt[5]{{\cdot}5742}$.

12. $\dfrac{1}{\sqrt{6{\cdot}783}}$.

13. $\sqrt{\dfrac{423}{578}}$.

14. $\dfrac{172\sqrt{47}}{287}$.

15. $483 \times ({\cdot}04172)^5$.

16. $\sqrt[3]{{\cdot}0176}$.

17. $\sqrt{\dfrac{{\cdot}1624}{{\cdot}0416}}$.

18. $(\cdot 00268)^{\frac{2}{3}} \times (\cdot 0246)^{\frac{1}{3}}$. 19. $(\cdot 01001)^{\frac{1}{4}}$.

20. Find the number of digits in 91^7.

21. Find the number of ciphers before the first significant digit in $(\frac{3}{14})^{10}$.

22. Obtain the square root of $\dfrac{\sqrt[3]{\cdot 0125} \times \sqrt{31\cdot 15}}{\cdot 00081}$.

23. Solve $(\frac{6}{11})^x = \frac{9}{16}$.

24. Find approximately the amount of £317 in 10 years at 3 % compound interest.

25. If the population of a town increases at the rate of 4 % each year, in how many years will the population be doubled?

26. Find the mean proportional between 14·76 and 35·82.

27. Calculate the surface and volume of a sphere of radius 13·27 ft., given that the surface is $4\pi r^2$, and volume is $\frac{4}{3}\pi r^3$. $(\pi = 3\cdot 142.)$

28. Evaluate $\sqrt{28\cdot 65 \times 14\cdot 35 \times 11\cdot 05 \times 3\cdot 25}$.

29. Find the product of 4·177, 0·04177, 0·0004177, 4177000, and find the square root of $(0\cdot 07346)^3$.

30. Knowing the number of pounds in a cubic inch of a substance, you can find the number of kilograms in cubic cm. by multiplying by $0\cdot 4536 \times (2\cdot 54)^{-3}$. Express this multiplier as a decimal to 3 places.

If steel weighs 488 lbs. per cubic foot, how many kilograms per cubic centimetre does it weigh?

31. Without using the tables find the characteristics of

(1) $\log_7 15914$, (2) $\log_8 0\cdot 00187$.

32. Obtain the value of

$$\frac{327\cdot 4 \times \sqrt[3]{0\cdot 0006}}{\sqrt{62\cdot 81}}.$$

33. Calculate, as accurately as your tables permit, the value of the fraction

$$\frac{1234 \times (2345)^2 \times (345\cdot 1)^3}{\sqrt[4]{45\cdot 12} \times \sqrt[5]{5\cdot 123}}.$$

34. Solve $3^{3x-1} = 2^{2x+1}$.

35. Find the values of $\log_{12} 432$, $\log_{20} \cdot 2164$.

36. The time of oscillation of a pendulum in secs. is given by

$$t=2\pi\sqrt{\frac{l}{g}};$$

find this if $\pi=3{\cdot}142$, $l=126{\cdot}2$ cms., $g=981$.

37. The reduction factor of a galvanometer is given by

$$k=\frac{r\mathrm{H}}{2\pi n};$$

find k when $\pi=3{\cdot}142$, $r=16{\cdot}2$ cms., $\mathrm{H}={\cdot}18$, $n=5$.

38. Find the critical temperature of a gas given by the formula

$$\mathrm{T}=\frac{8}{27}\frac{a}{\mathrm{R}b},\ \text{when}\ \mathrm{R}=\frac{1{\cdot}0065}{273},\ a={\cdot}00874,\ b={\cdot}0023.$$

39. Calculate the velocity of sound in cms. per sec. from the formula

$$\mathrm{V}=\sqrt{\frac{\gamma p}{\rho}},\ \text{when}\ \gamma=1{\cdot}40,\ p=76\times13{\cdot}6\times981,\ \rho={\cdot}001293.$$

40. Find the temperature of a gas expanding adiabatically according to the formula $\mathrm{T}=273\times2^{\gamma-1}$, where $\gamma=1{\cdot}40$.

41. Find the wave-length of sodium light from the formula

$$\lambda=\frac{ax}{\mathrm{D}},\ \text{if}\ a={\cdot}2375\ \text{cm.},\ x={\cdot}089\ \text{cms.},\ \mathrm{D}=358\ \text{cms.}$$

42. Calculate (n) the modulus of torsion in a wire, given $n=\frac{81\,.\,l\,.\,\pi}{t^2a^4}$, where $l=144{\cdot}1$ cms., $t=4{\cdot}10$ secs., $a={\cdot}0625$ cms., $\mathrm{I}=\frac{6079\times(4{\cdot}325)^2}{2}$.

43. Find M, the viscosity of water, given $\mathrm{M}=\frac{\pi\mathrm{PR}^4t}{8\mathrm{LV}}$, when $\mathrm{P}=39{\cdot}25\times981$, $\mathrm{R}^2={\cdot}00788$, $t=47$ secs., $\mathrm{L}=23{\cdot}3$ cms., $\mathrm{V}=102{\cdot}5$ c.c.

44. Find the ratio of I_1 to I_2, given $\frac{\mathrm{I}_1}{\mathrm{I}_2}=\frac{t_1^2-t^2}{t_2^2-t^2}$, where $t_1=3{\cdot}81$ secs., $t_2=5{\cdot}19$ secs., $t=3{\cdot}26$ secs.

45. Find C, the capacity of a condenser from the formulae $\mathrm{C}=\frac{\mathrm{Q}}{\mathrm{E}}$, $\mathrm{Q}=\frac{\kappa\mathrm{T}\,.\,\mathrm{D}}{2\pi}\left(1+\frac{\lambda}{2}\right)$, given that $\mathrm{D}=1{\cdot}3$, $\log\kappa=\bar{7}{\cdot}3432$, $\mathrm{T}=6{\cdot}333$ secs., $\lambda={\cdot}425$, $\mathrm{E}=1{\cdot}08$.

46. Evaluate $\mathrm{Y}=\frac{mgl^3}{4bdt^3}$ (Young's modulus), when

$m=20$ grams, $l=38{\cdot}2$, $t={\cdot}32$ cms., $g=981$, $b=1{\cdot}287$ cms., $d={\cdot}00656$ cms.

53. Logarithms of Trigonometrical Functions.

The logarithms of the trigonometrical functions of acute angles are to be obtained from tables. As the characteristic cannot be seen by inspection it is printed as well as the mantissa. Also, to save confusion with regard to the sign of the characteristic, the number 10 is added in each case. The result is called the Tabular logarithm. In order to obtain the logarithm we mentally subtract 10 as we read the tables.

Thus in the table of Logarithmic Sines, we have the tabular logarithm of sine 68° 18′ is 9·9681, and hence

$$\log \sin 68^\circ\, 18' = \bar{1}\cdot 9681.$$

Note that the characteristic is printed once only, at the beginning of each line.

The same rules concerning the subtraction of differences for cosines, cotangent, cosecant hold good as in the tables of natural functions.

Examples. VII d.

1. Write down from the tables :—

$\log\sin 56^\circ\, 40'$, $\log\tan 27^\circ\, 13'$, $\log\sec 56^\circ\, 47'$, $\log\cos 43^\circ\, 26'$, $\log\cot 19^\circ\, 44'$, $\log\sin 123^\circ\, 15'$.

2. Find θ in each of the following cases :—

(1) $\log\sin\theta = \bar{1}\cdot 4762$. (2) $\log\cos\theta = \bar{1}\cdot 6254$.

(3) $\log\tan\theta = \cdot 5843$. (4) $\log\sec\theta = \cdot 8765$.

(5) $\log\tan\theta = \bar{1}\cdot 5843$.

3. Find the values of

(1) $\sin 43^\circ\, 12' \times \cos 28^\circ\, 17'$. (2) $\sin 130^\circ\, 15' \times \cos 120^\circ\, 3'$.

(3) $\dfrac{\tan 27^\circ\, 11'}{\operatorname{cosec} 56^\circ\, 23'}$.

4. If $\sin A = \dfrac{15\cdot 4 \sin 47^\circ\, 13'}{18\cdot 7}$, find two values of A less than 180°.

5. If $\cos\theta = \sqrt{\dfrac{28{\cdot}65 \times 14{\cdot}35}{17{\cdot}6 \times 25{\cdot}4}}$, find θ.

6. Given $a = \dfrac{356 \sin 37^\circ\ 16'}{\sin 63^\circ\ 27'}$, find a.

7. Obtain the value of $\dfrac{\sin 25^\circ \cos 37^\circ}{\tan 130^\circ}$.

8. The area of a Δ being $\frac{1}{2}ab \sin \mathrm{C}$, find the area where $a = 798$ ft., $b = 460$ ft., $\mathrm{C} = 55^\circ\ 2'$.

9. If $\tan\theta = \frac{619}{2737} \cot 28^\circ\ 54'$, find θ.

10. In a Δ, $\sin \mathrm{B} = \dfrac{b \sin \mathrm{C}}{c}$; find B when $b = 127{\cdot}3$ ft., $c = 59{\cdot}21$ ft., $\mathrm{C} = 27^\circ\ 22'$.

11. Find the value of $\dfrac{15a \cos l}{\mathrm{V}}$, the coefficient of diurnal aberration where a = radius of earth = 3960 mls., V = velocity of light = 186,000 mls. per sec., l = observer's latitude = $51^\circ\ 7'$.

12. The electric current in a wire is given by $\mathrm{C} = \dfrac{\mathrm{H}r \tan\phi}{2\pi}$; find its value when $\mathrm{H} = {\cdot}18$, $r = 16{\cdot}01$ cms., $\tan 2\phi = {\cdot}1723$, $\pi = 3{\cdot}142$.

13. The refractive index for glass is given by

$$\mu = \frac{\sin \dfrac{\delta + \theta}{2}}{\sin \dfrac{\theta}{2}}.$$

Find μ when $\delta = 43^\circ\ 51'$, $\theta = 64^\circ\ 54'$.

14. The coefficient of mutual induction being given by $\mathrm{M} = \dfrac{\mathrm{RQ}}{\mathrm{C}}$, where $\mathrm{Q} = \dfrac{2\pi}{t} \times 3{\cdot}6 \times 10^{-9}$ and $\mathrm{C} = \kappa \tan\delta$, find M when $\mathrm{R} = 400$ ohms, $t = 4{\cdot}5$ secs., $\kappa = 1{\cdot}3$, $\delta = 11^\circ$, $\pi = 3{\cdot}142$.

15. The strength of a magnetic field is given by the formula $\mathrm{H} = \pi n \sqrt{\dfrac{2\kappa}{r^3 \tan\theta}}$. Evaluate H when $\pi = 3{\cdot}142$, $n = {\cdot}42$, $\kappa = 274{\cdot}6$, $r = 26$, $\theta = 59^\circ\ 7'$.

16. In solving a triangle it was necessary to use the formulae $\cos\phi = \dfrac{2\sqrt{bc}\cos\frac{A}{2}}{b+c}$, $a=(b+c)\sin\phi$. Find a when $b=13{\cdot}2$ cms., $c=15{\cdot}6$ cms., $A=48^\circ\ 28'$.

17. Given that the force required to prevent a body slipping down a rough inclined plane of angle a is $\dfrac{W\sin(a-\lambda)}{\cos\lambda}$, where λ is the angle of friction. Find this force if $W=52$ grams weight, $a=32^\circ\ 14'$, $\lambda=15^\circ\ 20'$.

18. If $a=(b-c)\sec\phi$ where $\tan\phi = \dfrac{2\sqrt{bc}\sin\frac{A}{2}}{b-c}$, find a when $b=11{\cdot}64$ cms., $c=9{\cdot}38$ cms., $A=52^\circ\ 14'$.

19. Find H from the formula $H\tan\theta = \dfrac{4\pi nCa}{10(a^2+x^2)^{\frac{3}{2}}}$, where $n=25$, $a=13{\cdot}97$ cms., $\theta=20^\circ$, $C={\cdot}62$ amperes, $x=36{\cdot}1$ cms.

20. In a conical pendulum the angle the string makes with the vertical is given by $\cos\theta = \dfrac{g}{4n^2\pi^2 l}$; find θ if $g=32$, $n={\cdot}8$, $\pi=3{\cdot}142$, $l=11{\cdot}86$.

21. The number of minutes in the angle of deviation of the plumb line due to the earth's rotation being

$$\frac{180\times 60}{\pi}\,\frac{\omega^2 a\sin\lambda\cos\lambda}{g},$$

find the angle if $\omega=\dfrac{2\pi}{24\times 60\times 60}$, $a=4000\times 1760\times 3$, $g=32{\cdot}2$, $\lambda=52^\circ\ 4'$.

Miscellaneous Examples. E.

1. Find the angle of elevation of the sun when the shadow cast by a tower 200 ft. high is $12\frac{1}{2}$ ft. less than it was when the elevation of the sun was 27°.

2. Given $\cos A = \cdot 34$, find the value of $\tan \frac{A}{2}$ and explain the double answer.

3. If you had no book of tables and had to find out whether the following were approximately correct, state how you would do so, giving your working and reasoning:

$$\text{(i) } \log 3 = \cdot 5, \quad \text{(ii) the no. whose log is } -\tfrac{1}{4} \text{ is } \cdot 56,$$

$$\text{(iii) } \log \cdot 12 = 2 \log \cdot 35.$$

4. Find four angles between 0° and 360° which satisfy the equation

$$4 \tan \theta - 5 + \cot \theta = 0.$$

5. Two sides of a triangle are 13·6 cms. and 15·4 cms. and the included angle is 46°. What would be the increase in area if each of the two sides was lengthened by 0·3 cm.?

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6. I have two tables containing the logarithms of all numbers and the tabular logarithms of sines of all angles from 0° to 90° but I have no tabular logarithms of cosines or tangents. I want to find the tabular logarithm of the cosine and tangent of a certain angle, say 34° 27′. How am I to do so?

7. Evaluate $\dfrac{2\pi\kappa(t_2 - t_1)}{\log r_2 - \log r_1}$, where $\pi = 3\cdot 142$, $\kappa = 0\cdot 74$, $t_1 = 69\cdot 4$, $t_2 = 82\cdot 3$, $r_1 = 1\cdot 25$, $r_2 = 1\cdot 55$.

8. Two adjacent sides AB, AD of a parallelogram are 4″ and 5″ respectively. The diagonal AC is 7″. Calculate the angle BAD.

9. The line OC joining a point O on the circumference of a circle of radius a to the centre C, makes with OX, any line through O, an angle α. If r be the distance of any other point P on the circumference from O and θ the angle OP makes with OX, prove $r=2a\cos(\theta\sim\alpha)$.

10. A regular pentagon is inscribed within a circle of radius r; show that its perimeter is $10r\sin 36°$ and its area $5r^2\sin 36°\cos 36°$, and find its perimeter and area as nearly as the tables allow when $r=5''$.

---

11. One side of a right-angled triangle is 6·432 ft. long and the angle opposite to it is 37° 27′. Find (i) the area of the triangle, (ii) the length of the perpendicular from the right angle on the hypotenuse.

12. If a body is projected up an inclined plane of angle β, with a velocity V ft. per sec. making an angle α with the horizon, its range is $\dfrac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$. Find the range when $V=56{\cdot}4$, $\alpha=64°\ 10'$, $\beta=28°\ 16'$, $g=32{\cdot}2$.

13. The angle between two tangents of length a, from an external point to a circle of radius r, is θ. Prove by projection that $r=\dfrac{a(1-\cos\theta)}{\sin\theta}$, $a=\dfrac{r(1+\cos\theta)}{\sin\theta}$. If d be the distance from the external point to the centre of the circle, prove

$$d=a\cos\frac{\theta}{2}+r\sin\frac{\theta}{2}.$$

14. Find to the nearest tenth the positive value of x which satisfies $\dfrac{2x}{1-x^2}=\tan 12°$.

15. A ray of light after reflexion at a plane mirror makes with the perpendicular to the mirror at the point of incidence an angle equal to the angle it makes with this perpendicular at incidence. Prove that if the mirror is turned through an angle α the reflected ray will be turned through an angle 2α.

---

16. XB is the projection of AB on MN, the angle AXB being a right angle. Find the length of XB when $AB = 5$ inches and the angle ABX (a) is equal to 33°. If AB and BC are the sides of a square, and XB, BY their projections on MN, how must the square be placed for XY to have (i) the least, (ii) the greatest possible length, consistently with the conditions that B is always to be on MN and the square is to be above MN and in the same plane with it?

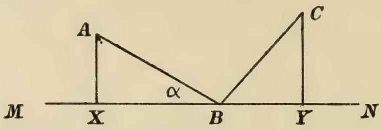

Fig. 55.

17. If a and β are two different angles which satisfy the equation $3+2\tan x=\sec x$, prove that $\tan(a+\beta)=\frac{12}{5}$.

18. An error of 1·5 % excess is made in measuring the side a of a triangle and of 1·8 % defect in measuring b. What is the resulting percentage error in the area as calculated from the formula $\frac{1}{2}ab\sin C$?

19. Find in acres the area of a triangular field, two of whose sides measure 576 and 430 yards, and meet at an angle of 54°.

20. A chord AB of a circle cuts a diameter CD at right angles at O. A line OE at right angles to the plane of the circle subtends at the points C, B, D angles of θ, a, ϕ respectively. Prove $\cot\phi=\cot^2 a \,.\, \tan\theta$.

CHAPTER VIII.

THE SOLUTION OF TRIANGLES.

54. The formula $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, proved in Art. 32, is not suitable for logarithmic work, but we can obtain from it formulae that are.

Thus we have

$$1+\cos A = 1+\frac{b^2+c^2-a^2}{2bc}$$

$$=\frac{b^2+2bc+c^2-a^2}{2bc}$$

$$=\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{2bc}.$$

Now let $$a+b+c=2s,$$

then $$b+c-a=2s-2a=2(s-a);$$

and $$\therefore\ 1+\cos A=\frac{2s\,.\,2(s-a)}{2bc};$$

$$\therefore\ 2\cos^2\frac{A}{2}=\frac{2s(s-a)}{bc};$$

$$\therefore\ \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}\ \ldots\ldots\ldots\ldots\ldots(1).$$

Explain why the positive root is taken in this result.

Similarly it can be shown that

$$1 - \cos A = 2\sin^2\frac{A}{2} = \frac{2(s-b)(s-c)}{bc};$$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \text{..............(2).}$$

From (1) and (2) we have

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \text{..............(3).}$$

Any one of these three formulae can be conveniently used for finding the angles of a triangle when the sides are given.

Example.

Find the angles of the triangle if $a=243{\cdot}4$, $b=147{\cdot}6$, $c=185{\cdot}2$.

$$\begin{aligned} a &= 243{\cdot}4 \\ b &= 147{\cdot}6 \\ c &= 185{\cdot}2 \\ 2)&\overline{576{\cdot}2} \\ s &= 288{\cdot}1 \\ s-a &= 44{\cdot}7 \\ s-b &= 140{\cdot}5 \\ s-c &= 102{\cdot}9 \end{aligned}$$

[A convenient test of accuracy $(s-a)+(s-b)+(s-c)=s.$]

$$\therefore \tan\frac{A}{2} = \sqrt{\frac{140{\cdot}5 \times 102{\cdot}9}{288{\cdot}1 \times 44{\cdot}7}};$$

$$\therefore \log\tan\frac{A}{2} = \tfrac{1}{2}\{\log 140{\cdot}5 + \log 102{\cdot}9 - \log 288{\cdot}1 - \log 44{\cdot}7\}$$

$$= {\cdot}0250;$$

$$\therefore \frac{A}{2} = 46^\circ\ 39';$$

$$\therefore A = 93^\circ\ 18'.$$

$$\begin{aligned} \log 140{\cdot}5 &= 2{\cdot}1476 \\ \log 102{\cdot}9 &= 2{\cdot}0123 \\ &\overline{4{\cdot}1599} \\ \log 288{\cdot}1 &= 2{\cdot}4596 \\ \log 44{\cdot}7 &= 1{\cdot}6503 \\ 2)&\overline{{\cdot}0500} \\ &{\cdot}0250 \end{aligned}$$

Also $$\tan\frac{B}{2}=\sqrt{\frac{44\cdot7\times102\cdot9}{288\cdot1\times140\cdot5}};$$

$$\log\tan\frac{B}{2}=\bar{1}\cdot5277;$$

$$\therefore\ \frac{B}{2}=18°\ 38';$$

$$\therefore\ B=37°\ 16';$$

$$\therefore\ A+B=130°\ 34';$$

$$\therefore\ C=49°\ 26'.$$

log 44·7	= 1·6503
log 102·9	= 2·0123
	3·6626
log 288·1	= 2·4596
log 140·5	= 2·1476
	2)$\bar{1}$·0554
	$\bar{1}$·5277

Note. We use the formula for the tangent here because we then only require to obtain four logarithms from the tables, viz. $\log s$, $\log(s-a)$, $\log(s-b)$, $\log(s-c)$.

To test accuracy we can find $\frac{C}{2}$ by the same method.

55. To solve a triangle when two sides and the included angle are given.

Let a, b, C be the given parts.

We have
$$\frac{\sin A}{\sin B}=\frac{a}{b};$$

$$\therefore\ \frac{\sin A-\sin B}{\sin A+\sin B}=\frac{a-b}{a+b};$$

$$\therefore\ \frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}=\frac{a-b}{a+b};$$

$$\therefore\ \tan\frac{A-B}{2}=\frac{a-b}{a+b}\tan\frac{A+B}{2}$$

$$\therefore\ \tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2},$$

since $$\frac{A+B}{2}=90°-\frac{C}{2}.$$

The above formula is suitable for logarithmic work, and from it we obtain the value of $\frac{A-B}{2}$.

And hence, since $\frac{A+B}{2}$ is known, we get the values of A and B. The side c can then be found, since

$$c=\frac{a \sin C}{\sin A}.$$

Example (i).

Given $b=253$, $c=189$, $A=72^\circ\ 14'$, solve the triangle.

First method.

We have $$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$=\tfrac{64}{442}\cot 36^\circ\ 7';$$

$$\therefore\ \log\tan\frac{B-C}{2}=\log 64-\log 442+\log\cot 36^\circ\ 7'$$

$$=\bar{1}\cdot2976;$$

$$\therefore\ \frac{B-C}{2}=11^\circ\ 13';$$

we have $\frac{B+C}{2}=53^\circ\ 53'$.

$$\begin{array}{rr}
\log\ 64= & 1\cdot8062\\
\log 442= & 2\cdot6454\\
 & \bar{1}\cdot1608\\
\log\cot 36^\circ\ 7'= & \cdot1368\\
 & \bar{1}\cdot2976
\end{array}$$

By addition $B=65^\circ\ 6'$.

By subtraction $C=42^\circ\ 40'$.

Also $$a=\frac{c\sin A}{\sin C}=\frac{189\sin 72^\circ\ 14'}{\sin 42^\circ\ 40'};$$

$$\therefore\ \log a=\log 189+\log\sin 72^\circ\ 14'-\log\sin 42^\circ\ 40'$$

$$=2\cdot4242;$$

$$\therefore\ a=265\cdot6.$$

$$\begin{array}{rr}
\log 189= & 2\cdot2765\\
\log\sin 72^\circ\ 14'= & \bar{1}\cdot9788\\
 & 2\cdot2553\\
\log\sin 42^\circ\ 40'= & \bar{1}\cdot8311\\
 & 2\cdot4242
\end{array}$$

Second method.

The following method does not involve the use of any special formula, and may sometimes be of use, but the results are likely to be less accurate than those obtained by the first method.

Let BD be perpendicular to AC.

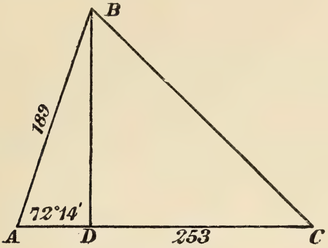

Fig. 56.

Then $AD = 189 \cos 72°\ 14'$;

$\therefore \log AD = 1{\cdot}7610$;

$\therefore AD = 57{\cdot}68$;

$\therefore CD = 195{\cdot}32$.

$$\begin{aligned} \log 189 &= 2{\cdot}2765 \\ \log \cos 72°\ 14' &= \bar{1}{\cdot}4845 \\ &\overline{\ \bar{1}{\cdot}7610\ } \end{aligned}$$

Also $BD = 189 \sin 72°\ 14'$;

$\therefore \log BD = 2{\cdot}2553$;

$\therefore BD = 180{\cdot}0$.

$$\begin{aligned} \log 189 &= 2{\cdot}2765 \\ \log \sin 72°\ 14' &= \bar{1}{\cdot}9788 \\ &\overline{\ 2{\cdot}2553\ } \end{aligned}$$

$$\tan C = \frac{BD}{CD} = \frac{180}{195{\cdot}3};$$

$\log \tan C = \bar{1}{\cdot}9646$;

$\therefore C = 42°\ 40'$.

$$\begin{aligned} \log 180 &= 2{\cdot}2553 \\ \log 195{\cdot}3 &= 2{\cdot}2907 \\ &\overline{\ \bar{1}{\cdot}9646\ } \end{aligned}$$

The rest of the solution is the same as in the first method.

Example (ii).

Given $a = 324$, $b = 287$, $B = 34°\ 17'$, solve the triangle.

We have $\sin A = \dfrac{a \sin B}{b} = \dfrac{324 \sin 34°\ 17'}{287}$;

$\therefore \log \sin A = \log 324 - \log 287 + \log \sin 34°\ 17'$

$= \bar{1}{\cdot}8033$;

$\therefore A = 39°\ 28'$;

or $140°\ 32'$.

Since $b < a$, both values of A are possible, and we have an ambiguous case. [Art. 35.]

$$\begin{aligned} \log 324 &= 2{\cdot}5105 \\ \log 287 &= 2{\cdot}4579 \\ &\overline{\ {\cdot}0526\ } \\ \log \sin 34°\ 17' &= \bar{1}{\cdot}7507 \\ &\overline{\ \bar{1}{\cdot}8033\ } \end{aligned}$$

(1) When
$$\mathrm{A} = 39^\circ\ 28';$$
$$\mathrm{A}+\mathrm{B} = 73^\circ\ 45';$$
$$\therefore\ \mathrm{C} = 106^\circ\ 15';$$

and
$$c = \frac{b \sin \mathrm{C}}{\sin \mathrm{B}} = \frac{287 \sin 106^\circ\ 15'}{\sin 34^\circ\ 17'} = \frac{287 \sin 73^\circ\ 45'}{\sin 34^\circ\ 17'};$$

$$\therefore\ \log c = 2\cdot6895;$$
$$\therefore\ c = 489\cdot3.$$

$$\log 287 = 2\cdot4579$$
$$\log \sin 34^\circ\ 17' = \bar{1}\cdot7507$$
$$2\cdot7072$$
$$\log \sin 73^\circ\ 45' = \bar{1}\cdot9823$$
$$2\cdot6895$$

(2) When
$$\mathrm{A} = 140^\circ\ 32';$$
$$\mathrm{A}+\mathrm{B} = 174^\circ\ 49';$$
$$\therefore\ \mathrm{C} = 5^\circ\ 11';$$

and
$$c = \frac{287 \sin 5^\circ\ 11'}{\sin 34^\circ\ 17'};$$

$$\therefore\ \log c = 1\cdot6626;$$
$$\therefore\ c = 45\cdot98.$$

$$\log 287 - \log \sin 34^\circ\ 17' = 2\cdot7072$$
$$\log \sin\ 5^\circ\ 11' = \bar{2}\cdot9554$$
$$1\cdot6626$$

Examples. VIII a.

Solve the following triangles :

1. $a=56\cdot4,\ b=75\cdot7,\ c=107\cdot5.$
2. $\mathrm{A}=37^\circ\ 14',\ \mathrm{B}=65^\circ\ 15',\ c=83.$
3. $\mathrm{B}=75^\circ\ 27',\ \mathrm{C}=43^\circ\ 12',\ b=27\cdot8.$
4. $a=264,\ b=435,\ \mathrm{C}=81^\circ\ 25'.$
5. $b=14\cdot76,\ c=28\cdot47,\ \mathrm{C}=46^\circ\ 30'.$
6. $a=28,\ c=33,\ \mathrm{A}=36^\circ\ 24'.$
7. $\mathrm{A}=107^\circ,\ a=456,\ b=312.$
8. $a=345\cdot2,\ b=281\cdot7,\ c=261\cdot5.$
9. $\mathrm{B}=41^\circ\ 15',\ \mathrm{A}=103^\circ\ 7',\ c=3\cdot47.$
10. $\mathrm{B}=122^\circ,\ a=43\cdot56,\ c=51\cdot45.$
11. $\mathrm{A}=57^\circ\ 14',\ \mathrm{B}=83^\circ\ 35',\ b=3147.$
12. In a triangle ABC, $a=35$, $b=43$ and $\mathrm{C}=75^\circ\ 11'$, find the angles A and B.

13. Given $A=42°$, $a=141$, $b=172·5$, find all solutions of the triangle ABC.

14. If $a=447$, $c=341$, $C=37° 22'$, find the two values of B; and draw a figure showing the two triangles obtained.

15. A, B are two points on one bank of a straight river, distant from one another 649 yards; C is on the other bank, and the angles CAB, CBA are respectively $48° 31'$ and $75° 25'$; find the width of the river.

16. The angles A, B of a triangle are respectively $40° 30'$ and $45° 45'$, and the intervening side is 6 feet; find the smaller of the remaining sides.

17. Find the greatest angle of the triangle whose sides are 184, 425 and 541.

18. In a triangle ABC the angles B and C are found to be $49° 30'$ and $70° 30'$ respectively, and the side a is found to be 4·375 inches. Find A, b and c as accurately as the tables permit.

19. If $a=1000$ inches, $b=353$ inches, $B=20° 35'$, find the angles A and C, taking A to be obtuse.

20. From Bristol to Richmond is 99 miles. From Richmond to Nottingham is 112 miles. From Nottingham to Bristol is 122 miles. If Richmond is due E. of Bristol, find the bearing of Nottingham from Bristol to the nearest degree.

21. A man walking along a road due E. sees a fort 4 miles away in a direction E. 32° N. If the guns have a range of 3 miles, how far must he go before he is (i) within range, (ii) out of range again?

22. OABC is a quadrilateral in which $OA=12·5$ ft., $OC=11$ ft., $\angle AOB=27° 40'$, $\angle BOC=35° 25'$. Find the angle OAC, and hence the distance of the intersection of the diagonals from O.

23. A rock slope is inclined at 40° to a horizontal plane. A man stands 30 yards from the foot of the slope, on the horizontal plane through it, and notices that the slope subtends 20° at his eye. If his eye is 5 ft. above the horizontal plane, find the length of the slope.

24. A coastguard travelling due E. along a straight coast road notices that, when he is due S. of a certain lighthouse, a distant church appears to be S.W., but a mile further on the lighthouse is N.W. by W. of him and the church W.S.W. Find the distance of the lighthouse from the road and from the church.

25. A ship runs from A 12·5 miles N. 40° W., then 14·6 miles N. 48° 11′ E., then 17·0 miles N.W. to B. Find how far she is N. of A and how far W. of A. Hence find the distance AB and the bearing of B from A.

26. A ship is sailing due W. at 10 knots. At midday a lighthouse bears N. 50° W. and at 12.15 it bears N. 30° W. When will its bearing be N. 35° E.?

27. A and B are two consecutive milestones on a straight road running N. From A a spire is observed to be N. 42° 17′ W. and from B the spire's bearing is N. 62° 13′ W. Find the shortest distance from the spire to the road.

28. B is 300 yds. from A along a road which runs N.E. From A a church tower bears N. 15° 43′ E. and from B its bearing is N. 12° 17′ W. Find how far the tower is from the road.

29. Two ships leave port at the same time. One sails W. 15° S. at 8·4 knots and the other W. 33° 14′ N. at 7·3 knots. How many sea miles will they be apart after two hours?

30. A and B are two fixed marks 1000 m. apart. An observer at P finds that APB is 36°. He then moves 600 m. in a straight line towards B to a point Q and finds that AQB is 53°. B is inaccessible, but the observer notes that ABP is an acute angle. Find AP and BP. What would be the result if ABP were obtuse?

31. A and B are two lighthouses at opposite ends of a harbour, A being due W. of B. AB = 500 yds. An observer on a ship entering the harbour on a straight course S. 35° W. sights B in the direction S. 8° W. and at the same time observes that A is S. 53° W. Find the distances of the ship from A and B when she crosses the line AB.

32. From a ship at sea a rock and a headland are observed to be in a straight line which runs N. 18° E. After the ship has sailed 6 miles to the N.W., the rock is seen to be due E. and the headland N.E. Find the distance of the rock from the headland.

33. A, B, C are three points such that AB = 476 yds., AC = 513 yds., BC = 495 yds. An observer who is in the same line as BC observes that AB subtends an angle of 23° 10′ at his eye. Find his distance from A.

34. From A a ship sails 8·4 miles N. 11° E., then 6·2 miles N. 52° 16′ W., then 4 miles N.E. to B. Find the distance and bearing of B from A.

35. A person wishing to determine the length of an inaccessible wall, places himself due S. of one end and then due W. of the other, at such distances that the angle which the length of the wall subtends at each station is 30°. Find the length of the wall, if the distance between the stations is 120 yds.

36. A, B, C are three stations in a straight line. AB = 1·75 miles, BC = 2·5 miles. I go from C to a station D, where I observe that BC subtends an angle of 23° 19′ and AB an angle of 17° 49′. Find AD and DB.

56. Frequently by the use of a subsidiary angle expressions may be thrown into a form suitable for logarithmic work.

Thus

$$\begin{aligned} a\sin\theta + b\cos\theta &= a\left(\sin\theta + \frac{b}{a}\cos\theta\right) \\ &= a(\sin\theta + \tan\alpha\cos\theta), \quad \text{where } \tan\alpha = \frac{b}{a} \\ &= \frac{a}{\cos\alpha}(\sin\theta\cos\alpha + \cos\theta\sin\alpha) \\ &= a\sin(\theta+\alpha)\sec\alpha. \end{aligned}$$

Here, by the use of the subsidiary angle α, we have thrown the expression $a\sin\theta + b\cos\theta$ into a form suitable for logarithmic work.

57. Again the formula $c^2 = a^2 + b^2 - 2ab\cos C$ can be put in various forms suitable for logarithmic work with the help of subsidiary angles; so that when two sides and the included angle of a triangle are given the third side can be found without first finding the other angles.

We proceed to give an example of this.

We have

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= a^2 + b^2 - 2ab\left(2\cos^2\frac{C}{2} - 1\right) \\ &= (a+b)^2 - 4ab\cos^2\frac{C}{2} \\ &= (a+b)^2\left\{1 - \frac{4ab}{(a+b)^2}\cos^2\frac{C}{2}\right\}. \end{aligned}$$

Now since $4ab < (a+b)^2$ and $\cos\frac{C}{2} < 1$, we can find an acute angle θ such that

$$\sin\theta = \frac{2\sqrt{ab}}{a+b}\cos\frac{C}{2}.$$

We then have

$$c^2 = (a+b)^2\{1 - \sin^2\theta\}$$
$$= (a+b)^2\cos^2\theta;$$
$$\therefore\ c = (a+b)\cos\theta.$$

Example.

The sides of a triangle are 237 and 158, and the contained angle is 58° 40′. Find the value of the base, without previously determining the other angles.

If $$a=237,\ b=158,\ \mathrm{C}=58^\circ\,40',$$

we have $$c=(a+b)\cos\theta,$$

where $$\sin\theta=\frac{2\sqrt{ab}}{a+b}\cos\frac{\mathrm{C}}{2}=\frac{2\sqrt{237\times 158}}{395}\cos 29^\circ\,20'.$$

To find θ, we have

$$\log\sin\theta=\log 2+\tfrac{1}{2}(\log 237+\log 158)-\log 395+\log\cos 29^\circ\,20',$$

	$\log 237 = 2{\cdot}3747$
	$\log 158 = 2{\cdot}1987$
	$2\,)\,4{\cdot}5734$
	$2{\cdot}2867$
$=\bar{1}{\cdot}9316;$	$\log 2 = \ {\cdot}3010$
$\therefore\ \theta = 58^\circ\,41'.$	$\log\cos 29^\circ\,20' = \bar{1}{\cdot}9405$
	$2{\cdot}5282$
	$\log 395 = 2{\cdot}5966$
	$\bar{1}{\cdot}9316$
$\therefore\ c = 395\cos 58^\circ\,41';$	$\log 395 = 2{\cdot}5966$
$\therefore\ \log c = 2{\cdot}3124;$	$\log\cos 58^\circ\,41' = \bar{1}{\cdot}7158$
$\therefore\ c = 205{\cdot}3.$	$2{\cdot}3124$

Examples. VIII b.

1. Show that $\sqrt{a^2+b^2}$ can be thrown into the form $a \sec\theta$, where $\theta = \tan^{-1}\frac{b}{a}$.

Give a geometrical interpretation to this by supposing a, b to be sides of a right-angled triangle.

2. Throw the expression $\dfrac{5\sin\theta + 3\cdot584\cos\theta}{5\sin\theta - 3\cdot584\cos\theta}$ into a form suitable to logarithmic calculation when different values of θ are introduced, and use your form to evaluate the expression when $\theta = 71°\,59'$.

3. In any triangle if $\tan\phi = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$, prove that

$$c = (a+b)\sin\frac{C}{2}\sec\phi.$$

Hence find c if $a=423$, $b=387$, $C=46°$.

4. Prove the formula

$$a^2 = \left(b+c-2\cos\frac{A}{2}\sqrt{bc}\right)\left(b+c+2\cos\frac{A}{2}\sqrt{bc}\right).$$

Apply it to find the side a of a triangle when $b=132\cdot5$ feet, $c=97\cdot32$ feet, $A=37°\,46'$, as accurately as the tables permit.

5. If ABC be a triangle, and θ such an angle that

$$\sin\theta = \frac{2\sqrt{ab}}{a+b}\cos\frac{C}{2},$$

find c in terms of a, b and θ.

If $a=11$, $b=25$ and $C=106°\,16'$, find c.

58. The area of a triangle in terms of the sides.

In Article 15 it was shown that $\Delta = \frac{1}{2}bc\sin A$.

Hence we have

$$\Delta = \frac{1}{2}bc \,.\, 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= bc \,.\, \sqrt{\frac{(s-b)(s-c)}{bc}} \,.\, \sqrt{\frac{s(s-a)}{bc}} \quad \text{[Art. 54]}$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

59. Radius of circumscribed circle.

From Article 15 Ex. (iii) we have

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C};$$

whence $$R = \frac{abc}{2bc \sin A}$$

$$= \frac{abc}{4\Delta}.$$

60. Radius of inscribed circle.

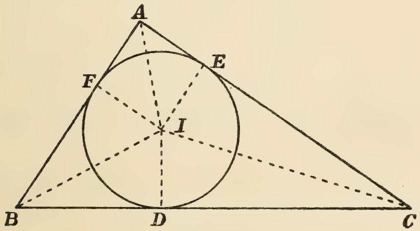

Fig. 57.

Let I be the centre of the inscribed circle of the triangle ABC, and D, E, F the points of contact with the sides. Let r be the radius. Then

$$\triangle ABC = \triangle BIC + \triangle CIA + \triangle AIB;$$

$$\therefore\ \Delta = \tfrac{1}{2}ra + \tfrac{1}{2}rb + \tfrac{1}{2}rc$$

$$= \tfrac{1}{2}r(a + b + c)$$

$$= rs;$$

$$\therefore\ r = \frac{\Delta}{s}.$$

61. Radii of escribed circles.

Let E be the centre of the escribed circle which touches BC and the other two sides produced.

Let P, Q, R be the points of contact, and r_1 the radius.

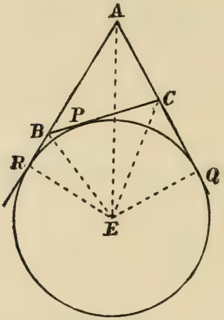

Fig. 58.

Then

$$\triangle ABC = \triangle EAC + \triangle EAB - \triangle EBC;$$

$$\therefore \Delta = \tfrac{1}{2}r_1 b + \tfrac{1}{2}r_1 c - \tfrac{1}{2}r_1 a$$

$$= \tfrac{1}{2}r_1 (b + c - a)$$

$$= \tfrac{1}{2}r_1 (2s - 2a)$$

$$= r_1 (s - a);$$

$$\therefore r_1 = \frac{\Delta}{s-a}.$$

Similarly the radii of the other escribed circles are

$$\frac{\Delta}{s-b}, \quad \frac{\Delta}{s-c}$$

62. There are many forms in which the radius of the inscribed circle may be expressed. Another form which is sometimes convenient can be obtained as follows.

Since tangents drawn from a point to a circle are equal, we have (Fig. 57)

$$\mathrm{BD} = \mathrm{BF}, \quad \mathrm{CE} = \mathrm{CD}, \quad \mathrm{AF} = \mathrm{AE};$$

$$\therefore \mathrm{BD} + \mathrm{CE} + \mathrm{AE} = \text{half the perimeter of the triangle}$$
$$= s;$$

$$\therefore \mathrm{BD} + b = s; \quad \therefore \mathrm{BD} = s - b.$$

Similarly $\quad \mathrm{CD} = s - c$, and $\mathrm{AE} = s - a$.

Hence we have

$$r = \mathrm{BD} \tan\frac{\mathrm{B}}{2} = (s-b)\tan\frac{\mathrm{B}}{2}$$

$$= (s-c)\tan\frac{\mathrm{C}}{2} = (s-a)\tan\frac{\mathrm{A}}{2} \text{ similarly.}$$

By combining this formula with $r = \dfrac{\Delta}{s}$ prove the formulae expressing $\tan\dfrac{\mathrm{A}}{2}$ etc. in terms of the sides of the triangle.

63. We can also obtain r_1 as follows, since (Fig. 58)

$$\mathrm{BR} = \mathrm{BP},$$
$$\mathrm{CQ} = \mathrm{CP},$$
and
$$\mathrm{AR} = \mathrm{AQ};$$

$$\therefore \mathrm{AR} = \tfrac{1}{2}(\mathrm{AR} + \mathrm{AQ}) = \tfrac{1}{2}(\mathrm{AB} + \mathrm{BP} + \mathrm{AC} + \mathrm{CP})$$
$$= \tfrac{1}{2}(a + b + c)$$
$$= s;$$

$$\therefore r_1 = \mathrm{AR}\tan\frac{\mathrm{A}}{2} = s\tan\frac{\mathrm{A}}{2}.$$

Similarly $\quad r_2 = s\tan\dfrac{\mathrm{B}}{2}, \quad r_3 = s\tan\dfrac{\mathrm{C}}{2}.$

Examples. VIII c.

1. Find correct to the tenth of a sq. inch the area of a triangle whose sides are 2·45, 3·17, 2·21 inches.

2. Find the radius of the inscribed circle of a triangle whose sides are 27·6, 13·8, 20·5.

3. A circle is circumscribed about a triangle whose sides are 17, 32, 43; find its radius.

4. A chord of a circle is 15·7 cm. in length, and the angle in one of the segments is 47°; what is the radius of the circle?

5. Find the radius of the largest circle which can be cut out of a triangle whose sides are 423, 375, 216 ft. Also calculate the area of the circle.

6. The lengths of the sides of a triangle are 375 links, 452 links, and 547 links. Find the length of the perpendicular upon the shortest side from the opposite corner, and the radius of the inscribed circle.

7. If the sides of a triangle are 17, 23, 30 inches in length, in what ratios do the points of contact of the inscribed circle divide them?

8. Prove that in an equilateral triangle the radii of the inscribed, circumscribed and escribed circles are as 1 : 2 : 3.

9. The sides of a triangle are 17, 25, 36; show that the radii of the escribed circles are as 21 : 33 : 154.

10. Prove that the radii of the inscribed and escribed circles can be expressed as

$$\frac{b \sin \frac{C}{2} \sin \frac{A}{2}}{\cos \frac{B}{2}}, \text{ and } \frac{b \cos \frac{C}{2} \cos \frac{A}{2}}{\cos \frac{B}{2}} \text{ respectively.}$$

11. Express the area of a triangle in terms of one side and the angles.

12. Prove that the distances between the centre of the inscribed circle and the centres of the escribed circles are

$$a \sec \frac{A}{2}, \quad b \sec \frac{B}{2}, \quad c \sec \frac{C}{2}.$$

Miscellaneous Examples. F.

1. The distances of a point P from two other points Q and R are wanted and cannot be directly measured. The distance between Q and R is found to be 1370 yds. $P\hat{Q}R = 33° 40'$, $P\hat{R}Q = 96° 25'$. Find the distances of P from Q and R, both by calculation and drawing.

2. If in the triangle ABC, C = 90°, prove

$$\cot \frac{A}{2} = \frac{b+c}{a}.$$

3. Calculate Young's Modulus from the formula $Y = \frac{4F \, . \, l^3}{bh^3 x}$, where $F = 500 \times 981$, $l = 70$, $b = 2{\cdot}22$, $h = 1{\cdot}28$, $x = 2$.

4. Two adjacent sides of a parallelogram are 6″ and 5″. Find the angle between them if the diagonal passing through their point of intersection is 9″.

5. Given that the diagonals of any quadrilateral are of length x, and y, and intersect at an angle θ, prove that the area of the figure is $\frac{1}{2}xy \sin \theta$.

---

6. The corner-post C of a property was fixed as being 87·6 chains from a tree and in the direction S. 56° 50′ E. This post having now been moved to a point C′ 25 chains due N. of C, the distance and direction of C′ from the tree must be determined. Find them by calculation.

7. A point P lies 3 miles from a point O in a direction 31° north of East; another point Q lies $5\frac{1}{2}$ miles from O in a direction E. 57° N. Calculate the distance between P and Q to the nearest tenth of a mile.

8. An isosceles triangle of vertical angle a is suspended by a string tied to its vertex and to an extremity of the base and rests so that the lower of the equal sides is horizontal. The angle made with the vertical by each portion of the string is θ and l is the length of the string, prove $l = \frac{h \cos a}{\sin \theta \cos \frac{a}{2}}$, where h is the altitude of the triangle.

9. Find θ from the formula $\cos\theta = \frac{g}{4n^2\pi^2 l}$, where $g=32$, $n=\cdot 84$, $\pi=3\cdot 142$, $l=11\cdot 8$.

10. Find the radius of a sphere of volume 320 c.c., given that volume $=\frac{4}{3}\pi r^3$. ($\pi=3\cdot 142$.)

---

11. In a triangle ABC, BC$=93$ yards, $A\hat{B}C=59° 19'$, $A\hat{C}B=43° 15'$. Calculate the length of AB.

Also find what error is made in the length of AB if the angle ACB is through a wrong measurement taken as $43° 17'$.

12. Find the number of years in which £320 will amount to £450 at 4 % Compound Interest.

13. A person on a cliff observes that the angles of depression of the light of a lightship 500 yds. away and its image by reflexion in water (which is the same distance vertically below the surface as the light is above) are D_1 and D_2, prove that the height of the cliff is $250(\tan D_1 + \tan D_2)$ yards.

14. In a triangle ABC, $a=25''$, $b=30''$, and $B=2A$, find the angles of the triangle and the third side.

15. Solve the equation $2^{x^2}=16^{x-1}$.

Find the number of digits in 19^{33}.

Find the number of zeros following the decimal point in the value of $(\frac{1}{19})^{33}$.

---

16. P and Q are two forts on the same side of a straight entrenchment. A base line XY of 1000 yards is measured along the entrenchment and the following angles are observed :—

$$Y\hat{X}P=95°,\quad X\hat{Y}P=43°,\quad X\hat{Y}Q=105°,\quad Q\hat{X}Y=27°.$$

Find the distance between the forts and check your result by drawing a plan to a scale of 6″ to a mile.

You may find useful the formula $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ or the formula $a=(b+c)\cos\phi$, where ϕ is given by

$$(b+c)\sin\phi = 2\sqrt{bc}\cos\frac{A}{2}.$$

17. An obtuse angled triangle has $a = 15 \cdot 3$ cms., $b = 9 \cdot 7$ cms., and $\hat{B} = 31^\circ 45'$. Calculate the remaining angles and draw the triangle accurately.

18. ABCD is a rectangular piece of paper having AB $= 14''$, BC $= 10''$. The paper is folded so that the corner C lies on AB and the crease makes 26° with the original position of the side CD. Calculate the length of the crease.

19. Prove that in any triangle

$$\tan \frac{B}{2} \tan \frac{C}{2} = \frac{s-a}{s}.$$

20. Find the volume of a regular tetrahedron (a pyramid, each face being an equilateral triangle) whose edge is $12''$ long.

Given, vol. of pyramid $= \frac{1}{3}$ area of base $\times$ altitude.

---

21. A rod AB, 3 feet long, is suspended by a string fastened to its two ends, which passes over a pulley at O, so that both portions of the string, OA and OB, make an angle of 20° with the vertical. If AB is inclined at 15° to the horizontal find the length of the string.

22. Express $\cos\theta + \sin\theta$ as the product of two cosines and hence find for what positive values of θ, less than 90°, the expression is (i) a maximum, (ii) a minimum.

23. If an error of 2 %₀ excess is made in measuring the sides a and b of a triangle, find the percentage error in the area calculated from the formula $\frac{1}{2}ab \sin C$.

24. When the sun is vertically overhead at the equator, an upright pole, 10 feet high, casts a shadow of 12 feet at a certain place. Find approximately the latitude of the place.

25. A is a point in the line XY. B and C are two points on the same side of XY. AB $= 4''$, AC $= 6''$, YAB $= 40^\circ$, BAC $= 60^\circ$. Calculate BC and find, by projecting on XY, the angle it makes with XY.

CHAPTER IX.

RADIAN OR CIRCULAR MEASURE OF ANGLES.

64. It may be either proved theoretically or verified by actual measurements that the circumference of a circle bears a constant ratio to the diameter.

This constant ratio is represented by the Greek letter π, so that $\dfrac{\text{circumference}}{\text{diameter}} = \pi$,

or circumference of a circle $= 2\pi R$ where R is the radius.

The value of π has been calculated to some 707 decimal places. For accurate results it may be taken as 3·14159 or 3·1416; but for rougher approximations $\frac{22}{7}$ (= 3·143), which is correct to two places, will be more useful:

$\frac{355}{113}$ gives 3·14159.

In working examples π is taken to be $\frac{22}{7}$ or 3·142 or 3·14159 according to the degree of accuracy required, and the answer must be given up as correct only to the number of significant figures justified by the data.

65. In theoretical investigations angles are not measured in degrees but in terms of a much more convenient unit called a *Radian*.

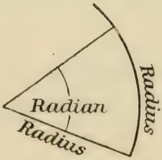

Fig. 59.

A Radian is the angle subtended at the centre of a circle by an ARC equal in length to the Radius.

It will be noticed that the angle subtended by a chord equal to the radius is 60°, so that a radian will be slightly less than 60°.

It will shortly be seen that the angle is of constant magnitude and in no way varies with the dimensions of the circle, otherwise of course it could not be used as a unit of measurement.

66. To measure any angle in terms of a Radian.

Let AOP be the angle.

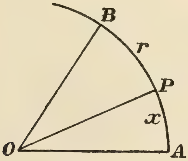

Fig. 60.

With centre O and any radius (r) draw a circle APB and suppose the arc AB $= r$, AP $= x$. Then

$$\angle \text{AOB} = 1 \text{ radian.}$$

Since angles at the centre of a circle are proportional to the arcs on which they stand, we have

$$\frac{\angle \text{AOP}}{1 \text{ radian}} = \frac{x}{r};$$

$$\therefore \text{ the number of radians in } \angle \text{AOP is } \frac{x}{r}.$$

Hence if θ be the number of radians in an angle which is subtended at the centre of a circle of radius r by an arc of length x, we have

$$\theta = \frac{x}{r},$$

$$\text{or} \quad \mathbf{x = r\theta}.$$

67. If the angle at the centre of the circle is 180°, we have

$$\frac{180^\circ}{1 \text{ radian}} = \frac{\text{semicircumference}}{r}$$
$$= \frac{\pi r}{r}$$
$$= \pi;$$
$$\therefore\ 180^\circ = \pi \text{ radians};$$
$$\therefore\ 1 \text{ radian} = \frac{180^\circ}{\pi}$$
$$= 57^\circ\ 17'\ 44'' \text{ approximately,}$$

and is therefore of constant magnitude.

It is important to remember that π denotes a *number*, namely, the ratio of the circumference of a circle to its diameter, which is approximately 3·1416; but it is usual to speak of "the angle π," meaning an angle of π radians, which is 180°.

Similarly "the angle $\frac{\pi}{3}$" means an angle of $\frac{\pi}{3}$ radians, which is 60°.

Example (i).

Express 20° 14′ in radian measure.

We have $20^\circ\ 14' = 20\tfrac{7}{30}{}^\circ,$

$$= \frac{20\tfrac{7}{30}}{180}\pi \text{ radians}$$
$$= \frac{607\pi}{5400} \text{ radians}$$
$$= \cdot 3532 \text{ radians.}$$

Example (ii).

Assuming the earth to be a sphere of 4000 miles radius, find the distance measured on the earth's surface between two places on the same meridian whose latitudes are 55° 16′ and 37° 40′.

Let A, B represent the two places, and C the point where the meridian through A and B meets the equator.

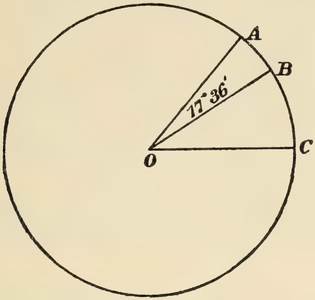

Fig. 61.

Then $\angle AOC =$ latitude of $A = 55° 16'$,

and $\angle BOC =$ latitude of $B = 37° 40'$;

$$\therefore \angle AOB = 17° 36'$$

$$= \frac{17\frac{3}{5}}{180}\pi \text{ radians};$$

$$\therefore \text{arc } AB = \frac{17 \cdot 6 \times \pi}{180} \times 4000 \text{ miles}$$

$$= 1230 \text{ miles approx.}$$

Or thus, from first principles

$$\frac{\text{arc } AB}{2\pi \times 4000} = \frac{17\frac{3}{5}°}{360°}.$$

Examples. IX a.

1. Express in radian measure as a fraction of π the angles $30°$, $150°$, $65°$, $74° 35'$.

2. Express in sexagesimal measure the angles whose radian measures are $\frac{\pi}{4}$, $\frac{2\pi}{3}$, $\frac{5\pi}{7}$, $\frac{5\pi}{3}$.

3. Find, to 2 places of decimals, the radian measures of $72° 15'$, $47° 24'$, $134° 13'$.

4. Express in sexagesimal measure, to the nearest minute, the angles 1·24, ·63 radians.

5. Find the length of the arc of a circle of 12 cm. radius, which subtends an angle of 40° at the centre. Answer to the nearest millimetre.

6. Find the number of radians in the angle subtended at the centre of a circle of radius 5 ft. by an arc 3 inches long.

7. Express in radians the angle turned through by the minute hand of a clock in 20 minutes.

8. An angle whose radian measure is ·45 is subtended at the centre of a circle by an arc 4 inches long; find the radius of the circle.

9. Find the number of degrees in the angle subtended at the centre of a circle of 10 cms. diameter by an arc of length 4 cms.

10. Express in degrees and in radians the angle of a regular figure of 8 sides.

11. The length of a degree of latitude on the earth's surface being $69\frac{1}{4}$ miles, find the radius of the earth.

12. A wheel makes 20 revolutions per second; how long will it take to turn through 5 radians?

13. The circumference of a circle is found by measurement to be 21·43 cms. with a possible error of 1 mm.; find its radius as accurately as this measurement justifies.

14. The distance between two places on the equator is 150 miles; find their difference in longitude. Take the radius of the earth to be 4000 miles, correct to two significant figures.

15. The driving wheel of a locomotive engine 6 ft. in diameter makes 3 revolutions in a second. Find approximately the number of miles the train passes over in an hour.

16. By considering regular hexagons inscribed in, and circumscribed about a circle, show that the ratio of the circumference of a circle to its diameter lies between 3 : 1 and $2\sqrt{3} : 1$.

17. Find the distance on the earth's surface between two places on the same meridian whose latitudes are 23° N. and 14° S. respectively; assuming the earth to be a sphere of 4000 miles radius, correct to 2 significant figures.

18. Two circles whose centres are A and B and radii 1·8 in. and 0·6 in. respectively are placed so as to touch one another externally at C. A line is drawn to touch the first circle at P and the second circle at Q. Calculate the lengths of the common tangent PQ and of the arcs PC, CQ.

19. A band is stretched tightly round two wheels of radii 1 ft. and 4 ft. respectively whose centres are 10 ft. apart. Find the total length of the band to the nearest inch.

68. Limiting values.

Let an arc BB′ of a circle subtend an angle of 2θ radians at the centre O.

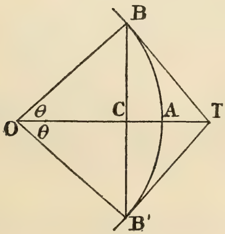

Fig. 62.

Draw BT, B′T the tangents at B and B′. Join BB′ and OT.

We shall assume that chord BB′ < arc BAB′ < BT + TB′.

(*Note.* A rigid proof that arc BAB′ < BT + TB′ is difficult and is beyond the scope of this book.)

Hence we have

$$\frac{BC}{OB} < \frac{\text{arc } BA}{OB} < \frac{BT}{OB};$$

i.e. $\sin\theta$, θ, $\tan\theta$

are in ascending order of magnitude.

Dividing by $\sin\theta$, we have

$$1, \quad \frac{\theta}{\sin\theta}, \quad \sec\theta$$

are in ascending order of magnitude.

Now as θ approaches the value zero, $\sec\theta$ approaches unity; $\therefore$ since $\frac{\theta}{\sin\theta}$ lies between 1 and $\sec\theta$, we have that the limiting value of $\frac{\theta}{\sin\theta}$, when $\theta \to 0$ is 1.

Using the notation of Art. 27, we have

$$\underset{\theta \to 0}{\mathrm{L}} \frac{\theta}{\sin\theta} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, by dividing $\sin\theta$, θ, $\tan\theta$ by $\tan\theta$, we have

$$\cos\theta, \quad \frac{\theta}{\tan\theta}, \quad 1$$

in ascending order of magnitude.

And as θ approaches zero, $\cos\theta$ approaches unity;

$$\therefore \quad \underset{\theta \to 0}{\mathrm{L}} \frac{\theta}{\tan\theta} = 1 \ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

From the results (1) and (2) we see that, if the angle is small, we may use its radian measure in place of its sine or tangent.

We may verify this by means of the tables.

Thus

radian measure of $3^\circ = \cdot 0524$,
$\sin 3^\circ = \cdot 0523$,
$\tan 3^\circ = \cdot 0524$.

For still smaller angles the degree of accuracy may be estimated from the following extract from 7-figure tables:

$\sin 10' = \cdot 0029089$,
$\tan 10' = \cdot 0029089$,
radian measure of $10' = \cdot 0029089$;
$\sin 19' = \cdot 0055268$,
$\tan 19' = \cdot 0055269$,
radian measure of $19' = \cdot 0055269$.

69. To find the area of a circle.

Suppose a regular polygon ABC of n sides to be inscribed in the ⊙, and one of n sides A′B′C′ to be described about the ⊙.

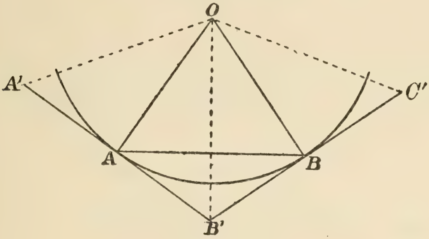

Fig. 63.

Then area of inscribed polygon

$$= n \,.\, \tfrac{1}{2}\mathrm{OA}\,.\,\mathrm{OB}\sin \mathrm{AOB}$$

$$= \frac{n}{2} r^2 \sin \frac{2\pi}{n}.$$

Area of circumscribed polygon

$$= n \,.\, \tfrac{1}{2}\mathrm{A'B'}\,.\,\mathrm{OA}$$

$$= n \,.\, \mathrm{AA'}\,.\,\mathrm{OA}$$

$$= n \,.\, \mathrm{AO}\tan\frac{\pi}{n}\,.\,\mathrm{OA}$$

$$= nr^2 \tan\frac{\pi}{n}.$$

The area of the circle lies between these values however great the number of sides may be.

Now when n is made infinitely great

$$\underset{n\to\infty}{\mathrm{L}} \frac{n}{2} r^2 \sin\frac{2\pi}{n} = \underset{n\to\infty}{\mathrm{L}} \frac{nr^2}{2} \,.\, \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} \,.\, \frac{2\pi}{n} = \underset{n\to\infty}{\mathrm{L}} \pi r^2 \,.\, \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}$$

$$= \pi r^2, \text{ by Art. 68,}$$

since $$\underset{n\to\infty}{\mathrm{L}} \frac{2\pi}{n} = 0.$$

Also

$$\underset{n \to \infty}{\mathrm{L}} \; nr^2 \tan\frac{\pi}{n} = \underset{n \to \infty}{\mathrm{L}} \; nr^2 \,.\, \frac{\pi}{n} \,.\, \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$$

$$= \underset{n \to \infty}{\mathrm{L}} \; \pi r^2 \,.\, \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}} = \pi r^2 ;$$

$$\therefore \text{ area of circle} = \boldsymbol{\pi r^2}.$$

70. Area of a sector of a circle.

If a sector of a circle contain an angle of θ radians at the centre, since sectors are proportional to the angles they contain, we have :

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta \text{ radians}}{2\pi \text{ radians}} ;$$

$$\therefore \text{ area of sector} = \frac{\theta}{2\pi} \,.\, \pi r^2 = \frac{\theta r^2}{2} .$$

This result may be written $\frac{1}{2}r(\theta r) = \frac{1}{2}rx$ where x is the length of arc subtended by θ.

71. If a distant object subtend a small angle at the point of observation, we can find a formula connecting the radian measure of the angle, and the approximate length and distance of the object.

Let l, d be respectively the approximate length and distance of the object, and let θ be the radian measure of the object subtended. Then the relation between these three quantities is

$$l = d\theta.$$

If we consider the length of the object as the length of

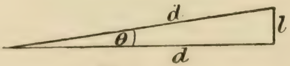

Fig. 64.

an arc of a circle of radius d, we have the above formula at once from Article 66.

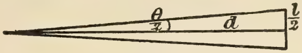

Fig. 65.

If we consider the length of the object as the base of an isosceles triangle of which the altitude is d, we have

$$\frac{l}{2} = d \tan \frac{\theta}{2}$$
$$= d\frac{\theta}{2},$$

since θ is small, Art. 68;

$$\therefore\ l = d\theta.$$

Example.

Given that the sun subtends an angle of $32'$ at a point on the earth's surface, and that the distance of the sun is 92×10^6 miles; find the sun's diameter.

The radian measure of $32' = \dfrac{32\pi}{60 \times 180}$;

$\therefore$ the diameter of the sun

$$= \frac{32\pi}{60 \times 180} \times 92 \times 10^6 \text{ miles approximately}$$

$$= \cdot 8563 \times 10^6$$
$$= 856000 \text{ miles.}$$

$\log 32 = 1\cdot 5051$
$\log \pi = \cdot 4972$
$\log 92 = 1\cdot 9638$
$3\cdot 9661$
$\log 60 = 1\cdot 7782$
$\log 180 = 2\cdot 2553$
$\bar{1}\cdot 9326 = \log \cdot 8563$

Note. Since the distance of the sun is only correct to two significant figures, we cannot rely on the above answer to more than two figures. Hence the result should be given as 860,000 miles. Also it should be remembered that results obtained by means of four figure tables cannot be expected to be accurate to more than three figures.

72. Dip of the Horizon.

Let ATB represent the earth, and O the position of an observer; then if tangents be drawn from O to the earth's

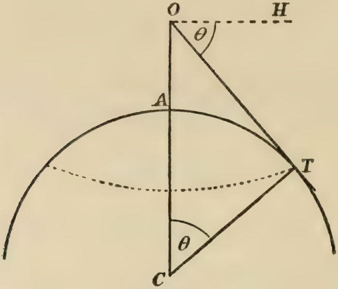

Fig. 66.

surface they will touch the earth in a circle, called the Visible Horizon.

If OH be the horizontal plane through O the angle HOT is called the Dip of the Horizon.

Ex. Find the dip of the horizon from a point 200 feet above sea-level, assuming the earth a sphere of radius 4000 miles.

From the figure $\angle$ HOT $=\angle$ TCO and $OT^2 = OA \cdot OB$, where B is the other extremity of the diameter. If r be the radius of the earth and h the length of OA in miles,

$$OT^2 = h(2r+h),$$

but since h is very small compared with r, h^2 is so small that it may be neglected;

$$\therefore \ OT = \sqrt{2rh}.$$

This is called the Distance of the Horizon.

Also since θ is a very small angle,

$$\theta \text{ radians} = \tan\theta = \frac{OT}{TC} = \frac{\sqrt{2rh}}{r} = \sqrt{\frac{2h}{r}};$$

$$\therefore \text{ the number of minutes in } \theta = \sqrt{\frac{2h}{r}} \times \frac{180}{\pi} \times 60$$

$$= \sqrt{\frac{2 \times 200}{4000 \times 1760 \times 3}} \times \frac{180 \times 60}{3{\cdot}142} = 14{\cdot}96'.$$

Examples. IX b.

1. Find the area of a circle of 10 inches radius.

2. Find the radius of a circle whose area is 426·24 sq. cms.

3. What is the area of a sector of a circle of radius 4 ft. which is bounded by two radii inclined at an angle of $60°$? Also find the area of the segment bounded by the chord joining the extremities of these radii.

4. The mean angular diameter of the moon being $31'$ when it is 240,000 miles away, find the diameter in miles.

5. If the sun is 93×10^6 miles distant, and subtends at the earth an angle of ·0093 radians, find its diameter.

6. Find the dip of the horizon from the top of a lighthouse 250 ft. high.

7. What is the distance of the visible horizon from the top of a cliff 300 ft. high?

8. Two lighthouses, each 200 ft. high, are so placed that the light of each is just visible from the other; what is the distance between the lighthouses?

9. From the formula $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, prove that if θ be an acute angle $\cos\theta$ lies between 1 and $1 - \frac{\theta^2}{2}$.

10. Deduce from the above result that $\sin\theta$ lies between θ and $\theta - \frac{\theta^3}{2}$.

11. Taking $\sin\theta = \theta$ (in radians) for a small angle, find $\sin 20'$ correct to three significant figures.

12. If θ be very small, prove that approximately

$$\sin(a+\theta) = \sin a + \theta\cos a,$$
$$\cos(a+\theta) = \cos a - \theta\sin a,$$
$$\tan(a+\theta) = \tan a + \theta\sec^2 a.$$

13. Prove that approximately the height of an object in feet is equal to $\dfrac{\text{distance in yards} \times \text{elevation in degrees}}{19}$.

14. Taking the diameter of a halfpenny to be 1 inch, find at what distance it will subtend $1°$ at the eye.

15. Find the perimeter and area of the crescent-shaped figure bounded by the arcs of two equal circles of radius 5 inches whose centres are 4 inches apart.

Miscellaneous Examples. G.

1. The latitude of London is $51°$ N. and the radius of the earth 4000 miles. How far is London from the equator measured along the earth's surface and how far from the earth's axis?

2. A man standing beside one milestone on a straight road observes that the foot of the next milestone is on a level with his eyes, and that its height subtends an angle of $2' \, 55''$. Find the approximate height of that milestone.

3. A rod ABC of length 7 ft. is held vertically at a point C on the side of a hill. From a point E at the foot the angle of elevation of A, the top of the rod, is $8° \, 12'$ and of B a point on the rod 3 ft. from the bottom the angle of elevation is $7° \, 18'$. Find the vertical height of C above E.

4. If D be the mid-point of BC in the triangle ABC, prove that

$$\cot CDA = \tfrac{1}{2}(\cot B - \cot C).$$

5. Two tangents are drawn to a circle of radius $4''$ from a point $10''$ from its centre. Find the lengths of the two arcs between the points of contact.

---

6. XAY is a straight line, AO a line 3 cms. long perpendicular to XAY, P is a point in XA, and the angle OPA is θ radians. With centre P and radius PO the circular arc OB is drawn to the line XAY and the tangent OC to this arc meets XAY in C. Suppose P to move continually away from A along AX and show what values the angle θ, the arc OB, the straight line OC, $\dfrac{\sin\theta}{\theta}$, $\dfrac{\tan\theta}{\theta}$, approach as P moves away.

Express $5°$ in radians, and compare it with the values of $\sin 5°$ and $\tan 5$ given by the tables.

7. How many miles an hour does London move in consequence of the rotation of the earth? Take the earth as a sphere of radius 3960 mls. London is in latitude $51° \, 30'$ N.

8. A man is on the perimeter of a circular space, and wishing to know its diameter, he selects two points in the boundary a furlong apart, which at a third point also in the boundary, subtend an angle of 164° 43′. Find the diameter to the nearest foot.

9. Find the radius of a sphere whose volume is 216·8 c.c., given volume $=\frac{4}{3}\pi r^3$, $\pi=3{\cdot}142$.

10. What is the distance of the visible horizon from the mast of a ship 80 feet high?

---

11. From a quadrant AB of a circle an arc AP is marked off subtending an angle of $x°$ at the centre. A circle with centre A passes through P and cuts the chord AB in P′. Express AP′ in terms of x. Suppose the chord graduated so that every point P′ corresponding to an integral value of x is marked x. How could you from a ruler graduated like this chord construct an angle of given magnitude?

12. Show that if an object of height h at a distance d from the observer subtends a small angle of A degrees at his position, then roughly $h=\frac{\mathrm{A}d}{57{\cdot}3}$. Use this to find the height of a tower which subtends an angle of 9° at a point 170 yards away.

13. A girder to carry a bridge is in the form of a circular arc: the length of the span is 120 ft. and the rise of the arch (i.e. the height of the middle above the ends) is 25 ft. Find the angle subtended by the arc at the centre of the circle and the radius of the circle.

14. Find the value of $({\cdot}03642)^{\frac{1}{3}} \times \cos 61° \, 23'$.

15. If the light from a lighthouse 250 ft. high can just be seen from the top of a mast 80 ft. high, find the approximate distance of the ship from the lighthouse, assuming the earth a sphere of 4000 mls. radius.

---

16. Taking $\sin\theta=\theta$ (in radians) for small angles, find sin 25′ correct to four significant figures.

17. Two places A and B on the earth's surface are on the same parallel of latitude 52° 30′. The difference of their longitudes is 32° 15′. Take the earth as a sphere of such size that a mile on the surface subtends an angle of 1′ at the centre, and find (i) the radius of the parallel of latitude on which A and B lie, (ii) the distance in a straight line between A and B, and (iii) the distance between A and B along a great circle, i.e. along a circle which passes through these points and has its centre at the centre of the earth.

18. A circle of radius r rolls on a horizontal straight line. A point P on the circle coincides with a point O on the straight line and after the circle has rotated through an angle θ the horizontal and vertical distances of P from O are x and y.

Prove $$x=r\theta-r\sin\theta,\quad y=r-r\cos\theta.$$

19. The figure is a rough sketch of a railway from A to B, which is made up of three straight pieces and two circular arcs. Calculate the length of the railway from A to B.

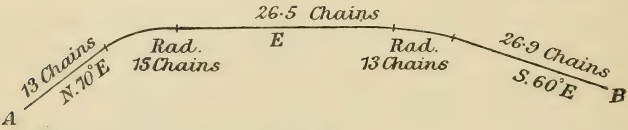

Fig. 67.

20. A chasm in level ground is bounded by parallel vertical sides. The depth AB of the chasm at A is wanted, and, it being impossible to take measurements from C, the point opposite A, a point D 50 yards along the side from C is chosen. The angle ADB is 43° and the angle ADC is 52°. Find the depth of AB.

---

21. The vertical angle A of a triangle ABC is bisected by AK which meets the base BC in K.

Prove $$\mathrm{AK}=\frac{2bc}{b+c}\cos\frac{\mathrm{A}}{2}.$$

22. On the base BC of a triangle ABC as hypotenuse, an isosceles right-angled triangle is described having its third vertex P on the same side of BC as A.

Prove that $$2\mathrm{PA}^2=b^2+c^2-2bc\sin\mathrm{A}.$$

23. Plot out the points A $(-2, -1)$, B $(1, 4)$, C $(4, 1)$.

Prove $$\tan BAC = \tfrac{6}{7}.$$

24. A straight ladder, 80 ft. long, rests with one end on horizontal ground, 60 ft. away from the wall of a house, and the other end on the roof of the house as high up as it can be. The wall of the house is 40 ft. high and the slope of the roof is 60° to the horizontal. Calculate the inclination of the ladder to the horizontal.

25. The three legs OA, OB, OC of a tripod are 4 ft. long. If they are supposed to meet at O and are placed with their other extremities on a horizontal plane, prove that O is vertically over the centre of the circumscribing circle of ABC. If the sides of ABC are 2 ft., 2·4 ft., and 1·8 ft., find the height of O above the plane ABC.

---

26. ABCD is a rectangle. $AB = a$, $BC = b$. B is joined to a point F in AD and DE is drawn perpendicular to BF produced. If $A\hat{B}F = \theta$, find by projection the length of DE and prove that the area of BDF is $\frac{1}{2}a(b - a\tan\theta)$.

27. Find x and y from the equations
$$x\cos\theta + y\sin\theta = a, \qquad x^2 + y^2 = a^2.$$

28. A cylinder of radius r is placed between two inclined planes sloping in opposite directions, the axis of the cylinder being parallel to the line of intersection of the planes. If the planes are inclined to the horizontal at angles α and β find the vertical height of the axis of the cylinder above the line of intersection of the planes.

29. In a certain triangle $\tan\frac{A}{2} = \frac{1}{2}$ and $\tan\frac{B}{2} = \frac{2}{3}$, find $\tan\frac{C}{2}$ and prove $a + b = 2c$.

30. Sketch the graph of the expression $\tan x - 2\sin x$ for values of x between 0° and 180° and hence solve approximately the equation $\tan x - 2\sin x = 0{\cdot}7$.

31. A vertical pole of height h subtends an angle a at a point O in a horizontal plane through the foot of the pole. Show that the vertical height that must be added to the pole so that the added portion also subtends an angle a at O is $h \sec 2a$.

32. Two circles, of radii a, b, touch each other externally: θ is the angle contained by the common tangents to these circles.

Prove that $$\sin \theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

33. Two points P, Q are taken on a circle of centre C and the chord PQ is drawn. Prove that if the larger of the two segments into which the circle is divided is 5 times the smaller, then $\theta - \sin \theta = \frac{\pi}{3}$.

34. O is the mid-point of the base AB of a triangle ABC and I is the centre of the inscribed circle of the triangle. ID is drawn perpendicular to AB. If $\mathrm{OD} = x$ and $\mathrm{DI} = y$ prove

$$\frac{4y^2}{c^2 - 4x^2} = \frac{s-c}{s}. \quad \left(\text{Use } \tan \frac{\mathrm{A}}{2}.\right)$$

35. Through the centre O of the circumscribing circle of a triangle ABC a straight line is drawn parallel to the side BC to meet the other two sides at D and E respectively. If $\mathrm{DE} = x$ prove that $$\frac{x}{a} = \frac{\cos(\mathrm{B}-\mathrm{C})}{2 \sin \mathrm{B} \sin \mathrm{C}}.$$

CHAPTER X.

ANGLES WHICH ARE NOT IN ONE PLANE.

73. We will begin by reminding the reader of some of the definitions and theorems of Solid Geometry.

(1) The intersection of two planes is a straight line.

(2) The angle between two planes is the angle between two straight lines drawn from any point in the line of intersection of the planes and perpendicular to it, one being in each plane.

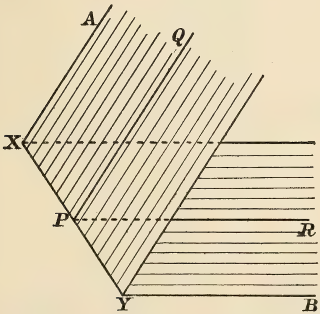

Fig. 68.

Thus in the figure, XY is the line of intersection of the two planes AXY, BXY.

Also if PQ, PR are both perpendicular to XY, and one of them lies in the plane AXY and the other in the plane BXY, then ∠ QPR is the angle between the planes.

(3) The angle a straight line makes with a plane is the angle between the straight line and its projection on the plane.

(4) If a straight line is perpendicular to each of two intersecting straight lines it is perpendicular to the plane which contains them; that is, it is perpendicular to every straight line in that plane which meets it.

(5) If N be the foot of the perpendicular from a point P to a plane, and Q be the foot of the perpendicular drawn from N to any straight line XY on the plane, then XY is perpendicular to the plane PNQ.

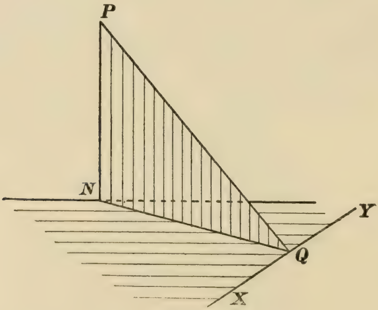

Fig. 69.

Thus in the figure, PN is perpendicular to every straight line which lies in the plane NXY and passes through N. NQ is the projection of PQ on the plane, and PQN is the angle of inclination of PQ to the plane. XY is perpendicular to the plane PNQ.

Example (i).

Suppose OX to be the intersection of a vertical with a horizontal plane.

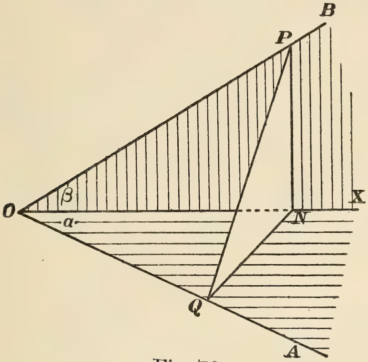

Fig. 70.

Let OA be in the horizontal plane making the angle a with OX; and let OB be in the vertical plane making the angle β with OX.

To find

(1) The angle AOB.

(2) The inclination of the plane AOB to the horizon.

From any point P in OB draw PN perpendicular to OX, and draw NQ perpendicular to OA.

Then PQ is perpendicular to OA. [Art. 73 (5).]

Now
$$\text{OQ} = \text{ON} \cos a$$
$$= \text{OP} \cos \beta \cos a;$$

$$\therefore \cos \angle \text{AOB} = \frac{\text{OQ}}{\text{OP}} = \cos a \cos \beta.$$

Again
$$\text{PN} = \text{ON} \tan \beta,$$
$$\text{QN} = \text{ON} \sin a.$$

Now the inclination of AOB to the horizon

$$= \angle \text{PQN}, \quad \text{[Art. 73 (2)]}$$

and we have
$$\tan \angle \text{PQN} = \frac{\text{PN}}{\text{QN}}$$
$$= \frac{\tan \beta}{\sin a}.$$

Example (ii)

A desk slopes at 15° to the horizon; find the inclination to the horizon of a line on the desk which makes 40° with the line of greatest slope.

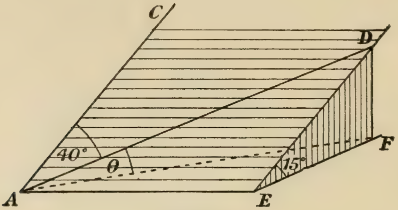

Fig. 71.

Let AB be the intersection of the plane of the desk with a horizontal plane. Also let AC be a line of greatest slope, and AD the line on the desk making 40° with AC. Take any point D in AD.

Draw DE parallel to AC, and DF perpendicular to the horizontal plane; then θ is the angle required.

Now $\angle \mathrm{DEF} = 15°$,

and we have $\mathrm{DF} = \mathrm{DE} \sin 15°$

$$= \mathrm{DA} \cos 40° \sin 15°,$$

since DEA is a right angle.

$$\therefore \sin\theta = \frac{\mathrm{DF}}{\mathrm{DA}}$$

$$= \cos 40° \sin 15°;$$

$$\therefore \log \sin\theta = \bar{1}\cdot 2973;$$

$$\therefore \theta = 11° \, 26'$$

approximately.

$$\log \cos 40° = \bar{1}\cdot 8843$$
$$\log \sin 15° = \bar{1}\cdot 4130$$
$$\overline{\bar{1}\cdot 2973}$$

Example (iii).

Two set squares, whose sides are 3, 4, 5 inches, are placed so that their shortest sides coincide, and the angle between the set squares is 40°. Find the angle between the longest sides.

Let ABC, ABD denote the set squares. We require the angle DAC.

Now $\angle$ CBD $=40°$; and if E be the middle point of CD, we have

$$CE = 4 \sin 20°;$$

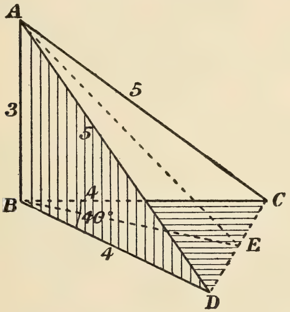

Fig. 72.

$$\therefore \sin CAE = \frac{CE}{CA}$$

$$= \frac{4 \sin 20°}{5} = \cdot 2736;$$

$$\therefore \angle CAE = 15° \; 53'$$

and $$\angle CAD = 31° \; 46' \text{ nearly.}$$

Example (iv).

The figure represents a rectangular box of which the sides are 3, 4, 5 feet.

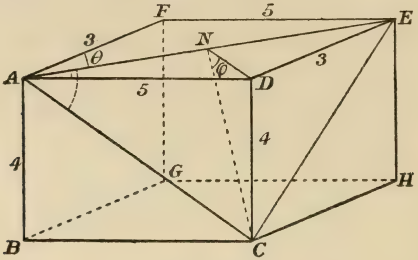

Fig. 73.

(1) The angle made by the plane ABHE with the plane ABGF

$$=\theta=\tan^{-1}\tfrac{5}{3}=\tan^{-1}1{\cdot}6667=59^\circ\ 2'.$$

(2) To find the angle between the planes AEC and ADEF; draw DN perpendicular to AE; then CN is also perpendicular to AE. Then ϕ is the angle required.

We have $\mathrm{DN}=\mathrm{DE}\sin\mathrm{DEA}=3\times\dfrac{5}{\sqrt{34}}=\dfrac{15}{\sqrt{34}}$, since $\mathrm{AE}=\sqrt{34}$;

$$\therefore\ \tan\phi=\frac{\mathrm{DC}}{\mathrm{DN}}=\frac{4\sqrt{34}}{15};$$

$$\therefore\ \phi=57^\circ\ 15'.$$

log 4 =	·6021
½ log 34 =	·7657
	1·3678
log 15 =	1·1761
	·1917

(3) To find the angle CAE, we have

$$\mathrm{CN}=\sqrt{\mathrm{CD}^2+\mathrm{DN}^2}=\sqrt{16+\frac{225}{34}}=\sqrt{\frac{769}{34}};$$

and $$\mathrm{CA}=\sqrt{16+25}=\sqrt{41};$$

$$\therefore\ \sin\mathrm{CAE}=\frac{\mathrm{CN}}{\mathrm{CA}}$$

$$=\sqrt{\frac{769}{34}}\div\sqrt{41}=\sqrt{\frac{769}{34\times 41}};$$

$$\therefore\ \angle\mathrm{CAE}=47^\circ\ 58'.$$

log 769 =	2·8859
log 34 =	1·5315
log 41 =	1·6128
2)	$\bar{1}$·7416
	$\bar{1}$·8708

Example (v).

To find the angle between two faces of a regular tetrahedron (i.e. a figure enclosed by four equal equilateral triangles).

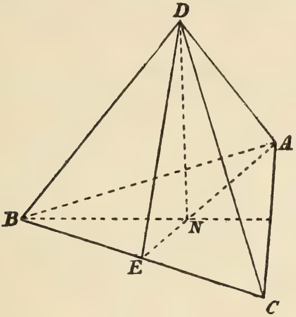

Fig. 74.

Let D, A, B, C be the vertices of the figure, and let a be the length of the side of each triangle.

Let E be the middle point of BC and N the foot of the perpendicular from D on the plane ABC.

Since DE is perpendicular to BC, and DN is perpendicular to the plane ABC,

∴ EN is perpendicular to BC at E the mid-point, and bisects the angle BAC. Similarly BN bisects the angle ABC.

∴ we have $\quad \mathrm{EN} = \mathrm{EB} \tan 30° = \dfrac{a}{2\sqrt{3}},$

and $\quad \mathrm{ED} = \mathrm{EB} \tan 60° = \dfrac{a\sqrt{3}}{2};$

$$\therefore \angle \mathrm{DEA} = \cos^{-1} \frac{\mathrm{EN}}{\mathrm{ED}} = \cos^{-1} \tfrac{1}{3}$$

$$= \cos^{-1} \cdot 3333$$

$$= 70° \; 32'.$$

And this is the angle between two faces.

Examples. X a.

1. Find the angle between a diagonal of a cube and a diagonal of one of the faces which meets it.

2. Find the angle between the diagonals of any two adjacent faces of a cube.

3. The edges of a rectangular box are 4, 3, 6 inches; find the length of a diagonal of the box, and the angle it makes with the longest side.

4. A triangle whose sides are as 3 : 4 : 5 is inclined to the horizon at an angle of 35°, and the longest side is horizontal. What are the inclinations of the other sides to the horizon?

5. A rectangle 6 ft. by 4 ft. is turned about the shorter side through an angle of 40°; find the angle between the two positions of one of the diagonals.

6. A desk slopes at 15° to the horizon and AB, the lower edge of it, is horizontal. A straight line AC is drawn on the desk making 35° with the lower edge and of length 20 inches. (1) How far is C from AB? (2) How far is C above the horizontal plane through AB? (3) What is the inclination of AC to the horizontal plane?

7. All the edges of a pyramid are of length a and its base is a square. Find the angle between one of the slant edges and the diagonal of the base which meets it. Find also the altitude of the figure.

8. A square of side 5″ rests on one edge and is inclined at an angle of 35° to the horizontal plane. Find the angle between a diagonal and its projection on the plane.

9. A rectangle 5 ft. by 4 ft. rests with its longer edge on a horizontal plane and is inclined at an angle of 52° to this plane. Find the length of the projection of a diagonal of this rectangle on the plane and the angle between the diagonal and its projection.

10. An isosceles triangle, base BC, 8″, equal sides AB, AC, 12″, rests with its base on a horizontal plane and is tilted over until it makes an angle of 40° with the plane. Find the height of the vertex above the plane and the angle between AC and its projection on the plane.

11. Two equal 45° set squares ABC, ABD are placed at right angles to one another and at right angles to a horizontal plane so that the edges AB coincide and B is on the plane. Find the angle the plane ACD makes with the horizontal plane, and the perpendicular distance of B from the plane ACD, if the shorter sides of the set squares are 5″.

12. Two vertical planes ZOX, ZOY inclined to one another at an angle of 20° intersect the horizontal plane in OX and OY. In the plane ZOY a point P is taken 8″ from OZ and 10″ from OY. Find the angle between the line OP and the plane ZOX.

13. Up a hillside sloping at 26° to the horizontal plane runs a zigzag path which makes an angle of 60° to the line of greatest slope. What is the length of the path to the top of the hill which is 1200 feet high and what angle does the path make with the horizontal plane?

14. O is a corner of a rectangular solid, and A, B, C are points on the three edges which meet at O. If OA, OB, OC are respectively 1, 2, 3 inches, find the angles the plane ABC makes with the faces of the solid.

15. Three straight lines OA, OB, OC are mutually at right angles, and their lengths are a, b, c. Show that the tangent of the angle between the planes OAB, ABC is $\dfrac{c\sqrt{a^2+b^2}}{ab}$, and hence that the area of $\triangle$ ABC is $\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}$.

16. A roof of a porch is built out at right angles to a vertical wall. The ridge AF is horizontal and of length 10 ft. The front face is an isosceles triangle FDE, whose edges FD, FE slope at 45° to the horizon, and the edge DE is 6 ft. The lower edges parallel to AF are each 14 ft. in length. Calculate the area of the roof.

17. XOY is the floor of a room; ZOX, ZOY are two vertical walls at right angles to one another. A stick AB rests with its end A on the floor 6 ft. from OX, and 3 ft. from OY. The other end B is fastened to the wall ZOY, 2 ft. from OY and 1 ft. from OZ. Find the length of the stick and of its projections on the walls ZOY, ZOX.

74. To find the height of a distant object.

Let AB denote the object, and let its height be h feet.

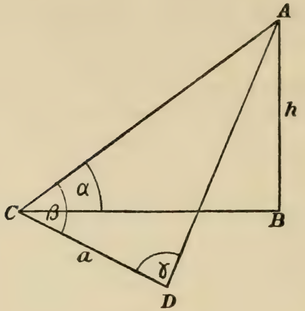

Fig. 75.

From a point C measure a straight line CD in any direction on a horizontal plane, and let its length be a feet. Let the angles ACB, ACD, ADC be observed to be α, β, γ respectively. Then we have

$$\text{AC} = h \operatorname{cosec} \alpha.$$

Also from the triangle ACD, we have

$$\frac{\text{AC}}{\sin\gamma} = \frac{\text{CD}}{\sin(180^\circ - \overline{\beta + \gamma})};$$

whence $$h \operatorname{cosec} \alpha = \frac{a \sin\gamma}{\sin(\beta + \gamma)};$$

$$\therefore\ h = \frac{a \sin\alpha \sin\gamma}{\sin(\beta + \gamma)}.$$

If the observations were made with a theodolite, the angles BCD, CDB would be observed instead of β and γ. In this case prove $h = \dfrac{a \tan\alpha \sin \text{CDB}}{\sin(\text{BCD} + \text{CDB})}$.

Example.

A man at A observes the angle of elevation of the top of a tower BC to be a.

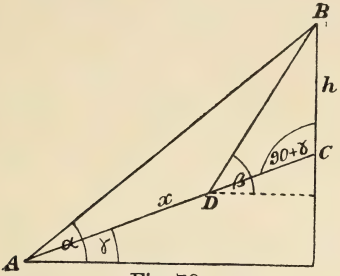

Fig. 76.

He walks x yards towards the tower up a road inclined at γ to the horizon and then observes the angle of elevation of B to be β. Find BC.

From the triangle BDC we have

$$\frac{\mathrm{BD}}{\sin(90+\gamma)}=\frac{h}{\sin(\beta-\gamma)}.$$

In the triangle ABD the angle ABD $=\beta-a$,

and
$$\frac{\mathrm{BD}}{\sin(a-\gamma)}=\frac{x}{\sin(\beta-a)};$$

$$\therefore\ \mathrm{BD}=\frac{x\sin(a-\gamma)}{\sin(\beta-a)};$$

$$\therefore\ h=\frac{x\sin(a-\gamma)}{\sin(\beta-a)}\,\frac{\sin(\beta-\gamma)}{\cos\gamma}.$$

Note that BD forms a connecting link between x and h.

In Art. 74 AC formed the connecting link between CD and h.

75. Projection of an area.

Let ABCD be a rectangle inclined at an angle θ to the horizon and having the side BC horizontal. Then if a, d are the projections of A and D on the horizontal plane, the

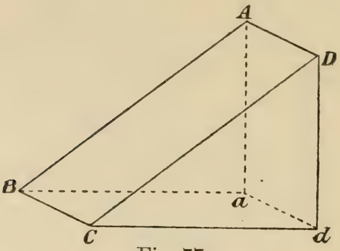

Fig. 77.

rectangle BadC is the projection of the rectangle ABCD; and the area of BadC is the area of ABCD multiplied by $\cos\theta$.

For $$\text{C}d = \text{CD}\cos\theta;$$

$\therefore$ area of
$$\begin{aligned}\text{B}ad\text{C} &= \text{BC} \times \text{C}d\\ &= \text{BC} \times \text{CD}\cos\theta\\ &= \text{area of ABCD} \times \cos\theta.\end{aligned}$$

It follows that if we have any figure of area A on a plane inclined to another plane XY at an angle θ, the area of the projection of the figure on the plane XY is equal to A $\cos\theta$.

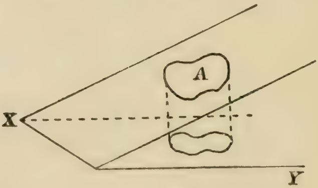

Fig. 78.

For the figure A may be considered to be composed of small rectangles having one side parallel to the line of section of the planes.

Examples. X b.

1. The elevation of a tower was observed at a certain station to be 25° and its bearing N.E. At a second station 1000 feet due S. of the former its bearing was N. by E. Find its height.

2. From a point A an observer finds that the angle of elevation of a peak BD is 37°. He walks 1000 yards to a point C on the same horizontal plane as A and D and observes the angles $BAC = 65°$, $ACB = 70°$. Find the height of the peak.

3. BC is a tower standing on a horizontal plane. From A and D two points in the plane 500 feet apart the angles of elevation of B, the top of the tower, are observed to be 20° 5′ and 27° 17′ respectively. The angle $CAD = 40°$. Find the height of the tower.

4. A ship was 2 miles due S. of a lighthouse. After sailing 1 mile W. 30° N. the angle of elevation of the top of the lighthouse was 2°. Find the height of the lighthouse above sea-level.

5. The angle of elevation of A the top of an inaccessible tower AB is observed from a point C to be 24°. A base line 400 ft. long is drawn from C to a point D and the angles BCD, CDB are observed to be 95°, 54° respectively. Find the height of the tower.

6. A lighthouse is seen N. 20° E. from a vessel sailing S. 30° E., and a mile further on it appears due N. Find its distance at the last observation.

7. A man at sea-level observes that the elevation of a mountain is 32° 11′: after walking directly towards it for a mile along a road inclined at an angle of 10° to the horizontal, he finds the elevation of the mountain to be 47° 23′. Find the height of the mountain.

8. From the top of a hill the depression of a point on the plain below is 40°, and from a place $\frac{3}{4}$ of the way down the depression of the same point is 20°. Find the inclination of the hill.

9. To find the distance of a battery B from a fort F, distances BA, AC were measured on the ground to points A and C, BA being 1000 yards and AC 1500 yards. The following angles were observed: $BAF = 33° 41'$, $FAC = 73° 35'$, $FCA = 81° 4'$. Find the distance BF.

10. From a certain station the angular elevation of a peak in the N.E. is observed to be 32°. A hill in the E.S.E. whose height above the station is known to be 1200 ft. is then ascended and the peak is now seen in the N. at an elevation of 20°. Find the height of its summit above the first station.

11. A balloon was observed in the N.E. at an elevation of 51° 50′: 10 minutes afterwards it was found to be due N. at an elevation of 31°. The rate at which the balloon was descending was afterwards found to be 6 miles per hour. Find the velocity of its horizontal motion (supposed uniform), the wind at the time being in the East.

12. A rectangular vertical target standing on a horizontal plane faces due S. Compare the area of the target with that of its shadow when the sun is S. 20° E. and at an altitude of 53°.

13. Find the height of a mountain whose summit is A, given that the length of a horizontal base line BC is 1500 yards, $\angle ABC = 61° 10'$, $\angle ACB = 52° 11'$, and the angle which AB makes with the vertical $= 57° 18'$.

14. A hill which slopes to the N. is observed from two points on the plane due S. at distances of 200 and 500 yards. If the angles of elevation of the top of the hill from these points are 32° and 25° respectively, find the inclination of the hill to the vertical.

15. From the top of a hill 1000 ft. above a lake the angle of elevation of a cloud is 21° 11′, and the angle of depression of its reflexion in the lake is 46° 3′. Find the height of the cloud.

16. A and B are two places 10 miles apart, B bearing E. 18° N. of A. A man is at P which bears S. 18° 36′ W. of A, and S. 52° 17′ W. of B. Find in what direction he must move to walk straight to a place Q 7 miles away from both A and B to the South of AB. Calculate also the distance from P to Q.

17. A seam of coal, 10 ft. thick, is inclined at 20° to the horizon. Find the volume of coal under an acre of land.

18. The area of the cross-section of a cylinder is 14·7 sq. ins. What is the area of a section making an angle of 10° with the cross-section?

19. A district in which the surface of the ground may be regarded as a sloping plane has an area of 5·8 sq. mls. It is shown on the map as an area of 4·6 sq. mls. At what angle is the plane inclined to the horizon?

20. A vertical wall 40 ft. long and 10 ft. high runs east and west; calculate the area of the shadow cast by it on the ground when the sun is S.S.W. at an elevation of 20°.

Trigonometrical Surveying.

76. Triangulation.

A district or country is surveyed by constructing a series of triangles, the sides of which are calculated from measurements of the various angles and the known length of one side of the initial triangle called the Base Line.

Angles in a horizontal or vertical plane are measured by an instrument called a Theodolite.

A survey which extends over a country large enough to necessitate the application of Spherical Trigonometry to allow for the curvature of the earth's surface is called a Geodetic Survey.

The Base Line for such a survey may be as much as 14 miles in length and is measured with great accuracy by a nickel-steel wire which has no coefficient of expansion for variations of temperature. Since the base line is not horizontal, the differences of level have to be measured and the observations reduced to sea-level.

For smaller triangulations the base line is measured with sufficient accuracy by a surveyor's chain, 22 yards long, consisting of 100 links.

The triangles observed should be as nearly equilateral as possible and small angles should be avoided as any error in their measurement would considerably affect the accuracy of the calculations.

If the angles of the triangle do not add up to 180° the difference between their sum and 180° is divided equally among them.

Example (i).

The base line CD was 8·895 chains.

At C the angles ECD, DCF were measured, also ECF and the re-entrant angle ECF, to check the observations.

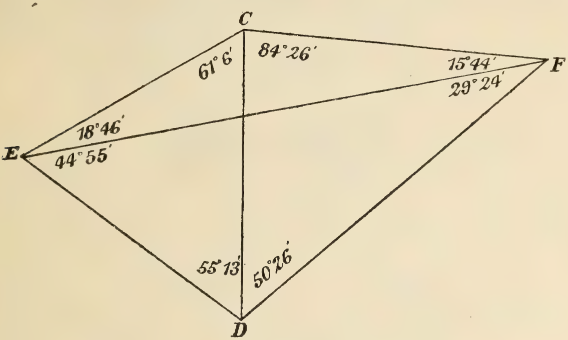

Fig. 79.

At D similar angles were observed. At E observations of C, F and D were made, and at F observations of C, E and D.

From the triangle ECD find EC, and from the triangle ECF find EF.

We have:

$$\frac{\text{EC}}{\sin 55^\circ 13'} = \frac{\text{CD}}{\sin 63^\circ 41'}; \quad \therefore \ \text{EC} = \frac{8{\cdot}895 \sin 55^\circ 13'}{\sin 63^\circ 41'}.$$

$$\frac{\text{EF}}{\sin 145^\circ 32'} = \frac{\text{EC}}{\sin 15^\circ 44'}; \quad \therefore \ \text{EF} = \frac{8{\cdot}895 \sin 55^\circ 13' \sin 34^\circ 28'}{\sin 63^\circ 41' \sin 15^\circ 44'}$$

$$= 17{\cdot}01 \text{ chains.}$$

Show that the same result is obtained by working with the triangle CDF to find DF and then with the triangle EFD to find EF.

Example (ii).

The diagram shows part of the triangulation of a river. When the principal triangulation is completed other points are fixed by using the sides of these triangles as base lines and the course of the river is determined by measurements of offsets from known points and lines.

Work with the triangles ABD and ADC to obtain

$$DC = 294{\cdot}45 \text{ ft.}$$

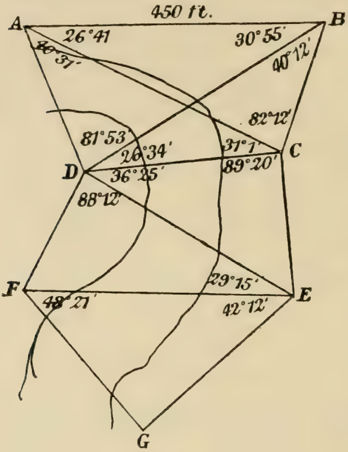

Fig. 80.

Then check by working with triangles ABC and BCD

$$DC = 294{\cdot}39 \text{ ft.}$$

Taking DC = 294·4 ft. work out the lengths of DE, FE, FG, GE.

In practice the length EG would be measured as a check base to confirm the accuracy of the observations and calculations.

Exercise.

The corners of a triangular field PQR are determined with reference to a base line AB by the dimensions PAB = 57°, PBA = 84°, QAB = 64°, QBA = 101°, RAB = 115°, RBA = 47°, AB is 50 feet long. Calculate the sides of the triangle PQR to the nearest foot.

Miscellaneous Examples. H.

1. Calculate the following by logarithms, and show how you would roughly check your results:

(1) pr^n, where $p=93{\cdot}75$, $r=1{\cdot}03$, $n=4$;

(2) $\frac{4}{3}\pi r^3$, where $\pi=\frac{355}{113}$, $r=5{\cdot}875$.

2. A man surveying a road from A to B, goes first 7 chains in a direction S. 63° E., then 8·3 chains S. 80° E., then 12 chains N. 46° E., and then 5·7 chains N. 16° W. to B. Find (1) how far B is east of A; (2) how far B is north of A; (3) the distance AB; (4) the bearing of B from A. Verify by a figure drawn to scale.

3. The sides of a quadrilateral taken in order are 4, 5, 8, 9 ft., and one diagonal is 9 ft.; find its angles and area.

4. ABCD is the rectangular floor of a room, the length BA being 48 ft. The height at C subtends at A an angle of 18°, and at B an angle of 30°. Find the height of the room.

5. In any triangle, prove

(1) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$;

(2) $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

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6. Calculate as accurately as the tables permit

(1) $\dfrac{52{\cdot}45\times 378{\cdot}4\times{\cdot}02086}{87{\cdot}32\times{\cdot}5844}$;

(2) $(1{\cdot}246)^{4{\cdot}195}$.

7. A ship sailing north sees two lighthouses which are 4 miles apart in a line due West. After sailing for an hour one of these bears S.W. and the other S.S.W. Find the ship's rate.

8. AB and DE are two chords of a circle at right angles to each other intersecting in C: AC = 40 ft., DC = 30 ft., and the radius of the circle is 100 ft. Find the sides and angles of the quadrilateral ADBE and determine its area.

9. If $\dfrac{\sin(\mathrm{C}-\theta)}{\sin\theta}=\dfrac{\cos\mathrm{A}}{\cos\mathrm{B}}$, where A, B, C are the angles of a triangle, prove that $\cot\theta=\tan\mathrm{B}$.

10. The plane side of a hill running E. to W. is inclined to the horizon at an angle of 20°: it is required to construct a straight railroad upon it inclined at 5° to the horizon. Determine the point of the compass to which it must be directed.

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11. A bed of coal 14 ft. thick is inclined at 23° to the surface. Calculate the number of tons of coal that lie under an acre of surface. A ton of coal occupies 28 c.ft. The 14 ft. is to be regarded as a measurement at right angles to the surface of the coal bed.

12. A pyramid of height 57 in. stands on a triangular base, one side of which is 25 in., the angles at the extremities of that side being 45° and 57° 30′. Find the volume to 2 significant figures.

13. The area of a triangle is 96 sq. ft. and the radii of the three escribed circles are 8, 12, 24 ft. respectively. Find the sides.

14. The angle of elevation of a tower 100 ft. high, and due N. of an observer is 50°. What will be its elevation to the observer when he has walked 300 ft. due E. of his former position?

15. If a, b, c are three consecutive integers, prove that

$$\log b-\log a>\log c-\log b.$$

---

16. If the sides of a triangle are 51, 68, 85 ft., show that the shortest side is divided by the point of contact of the inscribed circle into two segments, one of which is double of the other.

17. In a triangle ABC the side BC is 200 ft. long, and the angles at B and C are 79° and 75° respectively. B and C are observation stations and it is impossible to approach nearer to A. A body in the air h ft. immediately above A is observed to have an elevation of 40° at C. Calculate h.

18. OX, OY are two straight lines at right angles. On OX take a point P such that OP $=10$ cm. Now imagine OP to revolve to the position OY, and to vary in length in such a way that its length at any moment is equal to its original length multiplied by the cosine of the angle it has revolved through. Thus at 60° its length will be 5 cm. If x, y are the distances of P at any moment from OY, OX, show that $x^2+y^2-10x=0$.

19. In any triangle, prove that the area is equal to $\mathrm{R}r(\sin \mathrm{A}+\sin \mathrm{B}+\sin \mathrm{C})$, where R, r are the radii of the circumscribed and inscribed circles.

20. An upright pole 10 ft. high casts a shadow 12·6 ft. long at midday on a certain day. Another upright pole of the same height 100 miles further north casts a shadow 13·2 ft. long at the same time. Deduce the Earth's perimeter, supposing the Earth a sphere.

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21. A man whose eye is 5 ft. above the ground stands 20 ft. from the wall of a room, and observes the angle of elevation of one of the corners of the ceiling to be 30°. After walking 16 ft. directly towards the wall he finds the angle of elevation of the same corner to be now 60°. Find the height of the room.

22. A pole 15 ft. long leans against a wall with one end on the ground 9 ft. from the foot of the wall. This end is pulled away until the angle the pole makes with the ground is half what it was originally. Prove, without the use of tables, that the end is now $6\sqrt{5}$ ft. from the wall.

23. The side AB of a triangle ABC is divided at P in the ratio of $m : n$. The angles PCA, PCB, CPB are α, β, θ respectively. Prove that

$$m \cot \alpha - n \cot \beta = n \cot \mathrm{A} - m \cot \mathrm{B} = (m+n) \cot \theta.$$

24. Given
$$x = a \cos \theta + b \cos 2\theta,$$
$$y = a \sin \theta + b \sin 2\theta;$$
prove that
$$\cos \theta = \frac{x^2+y^2-a^2-b^2}{2ab}.$$

25. It was known to early Hindu mathematicians that if x, y and z are three angles such that $x-y=y-z=\mathrm{A}$, then $\sin x - \sin y = \sin y - \sin z + k \sin y$, and they used this formula to check tables of sines. Express k in terms of the angle A, and check your tables for the case $x=52°\ 24'$, $y=48°\ 42'$, $z=45°$.

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26. A man has before him on a level plain a conical hill of vertical angle 90°. Stationing himself at some distance from its foot he observes the angle of elevation α of an object which he knows to be half way up to the summit. Show that the part of the hill above the object subtends at his eye an angle

$$\tan^{-1}\frac{\tan\alpha(1-\tan\alpha)}{1+\tan\alpha(1+2\tan\alpha)}.$$

27. A rectangle ABCD in which $AB=b$, $BC=a$ is placed so that its diagonal AC, of length d, makes an acute angle ϕ with AX, a line passing through A. If AB makes an angle θ with AX, prove that

$$d\cos\phi=b\cos\theta-a\sin\theta,$$

and $$\tan\theta=\frac{b\tan\phi-a}{b+a\tan\phi}.$$

28. A straight bar 2 ft. long is suspended horizontally by two strings, each 2 ft. long, attached to its ends. The bar is twisted round its centre, the strings being kept tight, and the bar horizontal, till the centre is raised a foot. Through what angle is the bar twisted?

29. Two planes inclined at angles θ, ϕ to the horizon slope in opposite directions. A rod of length $2a$ making an angle α with the horizon rests with one end on each plane so that its mid-point is vertically over the line of intersection of the planes. Assuming that the line of intersection of the planes is horizontal, and that the rod lies in a vertical plane at right angles to this line, prove that $\tan\theta\sim\tan\phi=2\tan\alpha$.

30. An observer wishing to determine the length of an object in the horizontal plane through his eye, finds that the object subtends an angle α at his eye when he is in a certain position A. He then finds two other positions B, C where the object subtends the same angle α. Show that the length of the object is $\dfrac{abc\sin\alpha}{2\Delta}$, where a, b, c are the sides, and Δ the area of the triangle ABC.

APPENDIX.

IDENTITIES.

Chapter II.

Trigonometrical identities, being true for all values of the angles involved, are deduced from results established for all angles and should not be proved by reference to any particular figure.

To prove

$$\cot^2 A \tan A \sin A + \tan^2 A \cot A \cos A = \cos A + \sin A.$$

$$\cot^2 A \tan A \sin A + \tan^2 A \cot A \cos A$$

$$= \frac{\cos^2 A}{\sin^2 A} \cdot \frac{\sin A}{\cos A} \cdot \sin A + \frac{\sin^2 A}{\cos^2 A} \cdot \frac{\cos A}{\sin A} \cdot \cos A$$

$$= \cos A + \sin A.$$

Examples. II c.

Prove

1. $\sin A \sec A = \tan A$.
2. $\cos A \operatorname{cosec} A = \cot A$.
3. $\sin A \cot A = \cos A$.
4. $\sqrt{1 - \sin^2 A} \operatorname{cosec} A = \cot A$.
5. $\operatorname{cosec}^2 A + \sec^2 A = \operatorname{cosec}^2 A \sec^2 A$.
6. $\tan A + \cot A = \operatorname{cosec} A \sec A$.
7. $\cot^2 A - \cos^2 A = \cot^2 A \cos^2 A$.
8. $\cos A \sqrt{\sec^2 A - 1} = \sin A$.
9. $\cos^4 A - \sin^4 A = 1 - 2 \sin^2 A$.
10. $\cos^3 A - \sin^3 A = (\cos A - \sin A)(1 + \cos A \sin A)$.
11. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B$.
12. $(\sin A + \cos A)(\tan A + \cot A) = \sec A + \operatorname{cosec} A$.
13. $\dfrac{(\operatorname{cosec} A + \sec A)^2}{\operatorname{cosec}^2 A + \sec^2 A} = 1 + 2 \sin A \cos A$.
14. $\sin^4 \theta + \sin^2 \theta = 2 - 3 \cos^2 \theta + \cos^4 \theta$.

15. $\tan\theta+\cot\theta=\sqrt{\sec^2\theta+\operatorname{cosec}^2\theta}$.

16. $(\tan a-\sin a)^2+(1-\cos a)^2=(\sec a-1)^2$.

17. $(\cot A+\operatorname{cosec} A)^2=\dfrac{1+\cos A}{1-\cos A}$.

18. $\tan^2 A-\tan^2 B=\dfrac{\cos^2 B-\cos^2 A}{\cos^2 A\cos^2 B}$.

19. If $\sin x=a\sin y$ and $\tan x=b\tan y$, prove that

$$\cos x=\sqrt{\frac{a^2-1}{b^2-1}} \quad\text{and}\quad \cos y=\frac{b}{a}\sqrt{\frac{a^2-1}{b^2-1}}.$$

Chapter V.

Examples. V b.

In any triangle prove

1. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$.

2. $\dfrac{\sin A+\sin B}{\sin C}=\dfrac{a+b}{c}$.

3. $\cos A+\cos B=\dfrac{a+b}{c}(1-\cos C)$.

4. $\dfrac{a}{bc}+\dfrac{\cos A}{a}=\dfrac{b}{ca}+\dfrac{\cos B}{b}=\dfrac{c}{ab}+\dfrac{\cos C}{c}$.

5. $\dfrac{a\cos B-b\cos A}{c}=\dfrac{\sin^2 A-\sin^2 B}{\sin^2 C}$.

6. $2R(\sin^2 A+\sin^2 B+\sin^2 C)=a\sin A+b\sin B+c\sin C$.

7. $\dfrac{a^2+b^2-ab\cos C}{a\sin A+b\sin B+c\sin C}=\dfrac{a}{2\sin A}$.

8. $\dfrac{a\sin A+b\sin B+c\sin C}{ab\cos C+ac\cos B+bc\cos A}=\dfrac{1}{R}$.

9. If A, B, C and A, B′, C′ be the angles of two triangles having the same angle A and the same sides a, b, prove that

$$\frac{\sin C}{\sin B}+\frac{\sin C'}{\sin B'}=2\cos A.$$

10. If O is the centre of the circumscribing circle of the acute-angled triangle ABC and if AO produced meets BC in D, prove that

$$OD=\frac{R\cos A}{\cos(B\sim C)}.$$

Chapter VI.

In any triangle prove that

$$\cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0.$$

$\cos 2A + \cos 2B + \cos 2C$

$$= 2 \cos (A + B) \cos (A - B) + 2 \cos^2 C - 1$$
$$= -2 \cos C \cos (A - B) + 2 \cos^2 C - 1,$$
$$\text{since } \cos (A + B) = \cos (180^\circ - C),$$
$$= -2 \cos C \{\cos (A - B) - \cos C\} - 1$$
$$= -2 \cos C \{\cos (A-B) + \cos (A+B)\} - 1$$
$$= -2 \cos C \,.\, 2 \cos A \cos B - 1$$
$$= -4 \cos A \cos B \cos C - 1.$$

$\therefore \cos 2A + \cos 2B + \cos 2C + 4 \cos A \cos B \cos C + 1 = 0.$

Examples. VI g.

Prove

1. $\dfrac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A.$

2. $\cos^2 \dfrac{\alpha}{2} \left(1 + \tan \dfrac{\alpha}{2}\right)^2 = 1 + \sin \alpha.$

3. $\tan \alpha - \tan \beta = \sin (\alpha - \beta) \sec \alpha \sec \beta.$

4. $\sec \theta = 1 + \tan \theta \tan \dfrac{\theta}{2}.$

5. $2 \operatorname{cosec} 4\theta + 2 \cot 4\theta = \cot \theta - \tan \theta.$

6. $\cos 3A + \sin 3A = (\cos A - \sin A)(1 + 2 \sin 2A).$

7. $\cos \beta \cos (2\alpha - \beta) = \cos^2 \alpha - \sin^2 (\alpha - \beta).$

8. $\cos \alpha + \cos 3\alpha + \cos 5\alpha + \cos 7\alpha = \frac{1}{2} \sin 8\alpha \operatorname{cosec} \alpha.$

9. $\dfrac{\sin 3\theta + 2 \sin 5\theta + \sin 7\theta}{\sin 5\theta + 2 \sin 7\theta + \sin 9\theta} = \dfrac{\sin 5\theta}{\sin 7\theta}.$

10. $\dfrac{\sin x + \sin 2x + \sin 3x}{\cos x + \cos 2x + \cos 3x} = \tan 2x.$

11. $\dfrac{\sin A + 2 \sin 3A + \sin 5A}{\cos A - 2 \cos 3A + \cos 5A} = \dfrac{3 \operatorname{cosec} A - 4 \sin A}{4 \cos A - 3 \sec A}.$

12. $\sin 3\theta = 4\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta)$.

13. $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

14. $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4\cos 2\theta\cos 4\theta$.

15. $\cot\dfrac{a}{2} - \cot a = \operatorname{cosec} a$.

16. $\dfrac{1+\sin a}{1-\sin a} = \tan^2\left(45+\dfrac{a}{2}\right)$.

17. $\dfrac{\sin(A+B+C)}{\cos A\cos B\cos C} = \tan A + \tan B + \tan C - \tan A\tan B\tan C$.

18. $\dfrac{\sec B - \sec A}{\operatorname{cosec} A - \operatorname{cosec} B} = \tan A\tan B\tan\dfrac{A+B}{2}$.

19. $(x\tan a + y\cot a)(x\cot a + y\tan a) = (x+y)^2 + 4xy\cot^2 2a$.

20. $\sin(a-\beta)+\sin(\beta-\gamma)+\sin(\gamma-a)$

$$= -4\sin\frac{a-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-a}{2}.$$

21. $\cos^2 A + \cos^2 B + \cos^2(A+B) = 1 + 2\cos A\cos B\cos(A+B)$.

22. If $A+B+C=2S$, prove

$$\sin(B-C)\cos(S-A)+\sin(C-A)\cos(S-B)+\sin(A-B)\cos(S-C)=0.$$

23. If $a=\beta+\gamma$, prove

$$\sin(a+\beta+\gamma)+\sin(a+\beta-\gamma)+\sin(a-\beta+\gamma)=4\sin a\cos\beta\cos\gamma.$$

24. If $a+\beta+\gamma=0$, prove

$$\cos^2\beta-\cos^2\gamma-\cos^2 a = 1+2\sin\beta\sin\gamma\cos a.$$

25. $(\sec A + 2\sin A)(\operatorname{cosec} A - 2\cos A) = 2\cos 2A\cot 2A$.

26. If $\sin x = n\sin(x+2a)$, prove

$$\tan(x+a) = \frac{1+n}{1-n}\tan a.$$

27. $\dfrac{\sin(a-\gamma)}{\cos a\cos\gamma} + \dfrac{\sin(\beta-a)}{\cos\beta\cos a} + \dfrac{\sin(\gamma-\beta)}{\cos\gamma\cos\beta} = 0$.

28. $\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right) + \tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right) = 2\sec\theta$.

29. If $\cot x + \cot y = \operatorname{cosec} a\operatorname{cosec} y$, prove that

$$\sin x = \sin a\sin(x+y).$$

30. $\sec\theta - \tan\theta = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$.

31. $\sin^4\theta + 2\cos a \sin^2\theta\cos^2\theta + \cos^4\theta = 1 - \sin^2\frac{a}{2}\sin^2 2\theta$.

32. $\cos a + \cos\beta + \cos\gamma + \cos(a+\beta+\gamma)$

$$= 4\cos\frac{a+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+a}{2}.$$

33. If A, B, C are in A.P., prove that

$$\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{\cos B}{\sin B}.$$

34. $\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta$.

35. If $\cos a + \cos\beta + \cos\gamma = 0$, prove that

$$\cos 3a + \cos 3\beta + \cos 3\gamma = 12\cos a\cos\beta\cos\gamma.$$

36. If $$a\sin A + b\sin B + c\sin C = 0$$

and $$a\cos A + b\cos B + c\cos C = 0,$$

prove $$\frac{\sin(B-C)}{a} = \frac{\sin(C-A)}{b} = \frac{\sin(A-B)}{c}.$$

37. $\tan 5x - \tan 3x - \tan 2x = \tan 5x\tan 3x\tan 2x$.

38. If $A+B+C=2S$ and $\cos\theta = \frac{\cos A + \cos B\cos C}{\sin B\sin C}$,

prove $$\cos\frac{\theta}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}.$$

39. $\cos^3 2\theta + 3\cos 2\theta = 4(\cos^6\theta - \sin^6\theta)$.

40. If $\sin A + \cos B = \sqrt{2a+1}$ and $\cos A + \sin B = \sqrt{2b+1}$,

prove that $$\sin(A+B) = a+b.$$

41. $\cos^2 A - \cos A\cos(60^\circ + A) + \sin^2(30^\circ - A) = \frac{3}{4}$.

Examples. VI h.

Prove

1. $\cot^{-1}\frac{1}{3} - \cot^{-1}3 = \cot^{-1}\frac{3}{4}$.
2. $\tan^{-1}\frac{2}{11} + 2\tan^{-1}\frac{1}{7} = \tan^{-1}\frac{1}{2}$.
3. $\cos^{-1}[xy - \sqrt{1 - x^2 - y^2 + x^2y^2}] = \cos^{-1}x + \cos^{-1}y$.
4. $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$.
5. $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

6. If $$u=\cot^{-1}\sqrt{\cos a}-\tan^{-1}\sqrt{\cos a},$$
prove $$\sin u=\tan^2\frac{a}{2}.$$

7. $$\cos^{-1}\frac{1}{\sqrt{1+x^2}}-\cos^{-1}\frac{x}{\sqrt{1+x^2}}=\sin^{-1}\frac{x^2-1}{x^2+1}.$$

8. $$2\tan^{-1}\left(\tan\frac{a}{2}\tan\frac{x}{2}\right)=\cos^{-1}\left(\frac{\cos a+\cos x}{1+\cos a\cos x}\right).$$

9. If $$\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=a,$$
prove that $$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos a+\frac{y^2}{b^2}=\sin^2 a.$$

10. $$2\tan^{-1}\left(\sqrt{\frac{x-y}{x+y}}\tan\frac{a}{2}\right)=\cos^{-1}\frac{y+x\cos a}{x+y\cos a}.$$

Examples. VI k.

In any triangle prove that

1. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$

2. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

3. $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$

4. $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}.$

5. $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C.$

6. $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C.$

7. $\tan A+\tan B+\tan C=\tan A\tan B\tan C.$

8. $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C.$

9. If x, y, z be the perpendiculars drawn to the sides of a triangle ABC from the centre of the circumscribing circle, prove that
$$4\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)=\frac{abc}{xyz}.$$

10. If p, q are the perpendiculars from A and B on any line through C, prove that
$$a^2p^2+b^2q^2-2apbq\cos C=a^2b^2\sin^2 C.$$

Chapter VIII.

Examples. VIII d.

In any triangle prove that

1. $\dfrac{rr_1}{r_2r_3}=\tan^2\dfrac{A}{2}$.

2. $(a+b+c)^3\tan\dfrac{A}{2}\tan\dfrac{B}{2}\tan\dfrac{C}{2}=8r_1r_2r_3$.

3. $\dfrac{b-c}{a}\cos^2\dfrac{A}{2}+\dfrac{c-a}{b}\cos^2\dfrac{B}{2}+\dfrac{a-b}{c}\cos^2\dfrac{C}{2}=0$.

4. $r_1r_2+r_2r_3+r_3r_1=s^2$.

5. $a\cos^2\dfrac{B}{2}+b\cos^2\dfrac{A}{2}=b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}$

$$=c\cos^2\frac{A}{2}+a\cos^2\frac{C}{2}=s.$$

6. $\dfrac{r_1-r}{a}+\dfrac{r_2-r}{b}=\dfrac{c}{r_3}$.

7. $r=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

8. $2(R+r)=a\cot A+b\cot B+c\cot C$.

9. The sides of the triangle whose vertices are the centres of the escribed circles are $a\operatorname{cosec}\dfrac{A}{2}$, $b\operatorname{cosec}\dfrac{B}{2}$, $c\operatorname{cosec}\dfrac{C}{2}$.

10. The area of a triangle : the area of the inscribed circle

$$=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}:\pi.$$

11. $\sqrt{bc\sin B\sin C}=\dfrac{b^2\sin C+c^2\sin B}{b+c}$.

12. $8r\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}=R(\sin 2A+\sin 2B+\sin 2C)$.

13. The area of the triangle formed by joining the points of contact of the inscribed circle $=2r^2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.

14. If S, S' are the areas of a given triangle and the triangle formed by joining the points of contact of the inscribed circle, prove that

$$\frac{S'}{S}=\frac{2(s-a)(s-b)(s-c)}{abc}.$$

Miscellaneous Examples. K.

1. If $$a\cos\theta+b\sin\theta+c=0,$$
and $$a'\cos\theta+b'\sin\theta+c'=0,$$
prove that $$(bc'-b'c)^2+(ca'-c'a)^2=(ab'-a'b)^2.$$

2. If $\cos\alpha=\cos\beta\cos\theta$, prove that
$$\tan\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2}=\tan^2\frac{\theta}{2}.$$

3. If $\sin B\sin C=\cos^2\frac{A}{2}$, prove that the triangle ABC is isosceles.

4. If $\operatorname{cosec}\theta+\cot\theta=a$, prove that $\sec\theta+\tan\theta=\dfrac{a+1}{a-1}$.

5. If $x=c\tan\theta$, show that
$$(x^2+c^2)^2\sin 4\theta=4cx(c^2-x^2).$$

6. If $$x=a\cos\theta+b\cos 2\theta,$$
and $$y=a\sin\theta+b\sin 2\theta,$$
prove that $$a^2\{y^2+(x+b)^2\}=(x^2+y^2-b^2)^2.$$

7. If a, b, c, d be the sides of a quadrilateral and θ, ϕ, ψ be the angles respectively at which a and b, a and c, b and c are inclined to each other, prove
$$d^2=a^2+b^2+c^2-2ab\cos\theta-2ac\cos\phi-2bc\cos\psi.$$

8. If α and β be two acute angles, such that
$$3\sin^2\alpha+2\sin^2\beta=1,$$
and $$3\sin 2\alpha-2\sin 2\beta=0,$$
prove that $$\alpha+2\beta=90°.$$

9. If p, q, r are the perpendiculars on the sides BC, CA, AB of a triangle ABC from the opposite vertices, prove that
$$a\sin A+b\sin B+c\sin C=2(p\cos A+q\cos B+r\cos C).$$

10. If $x=\operatorname{cosec}\theta-\sin\theta$ and $y=\sec\theta-\cos\theta$,
prove that $$x^{\frac{2}{3}}y^{\frac{2}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=1.$$

ANSWERS.

I. p. 4.

1. 60°. 2. 52° 44′ 40″. 3. 37° 34′.
4. 150°, 210°. 5. 97° 30′, 262° 30′. 6. 1530°.
7. 98045. 8. 43° 44′ 24″.
9. (i) 120°, (ii) 128° 34′ 17″, (iii) 108°. 10. ·56995.

II a. p. 9.

1. BC$=3$, $\frac{3}{5}$, $\frac{3}{5}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{5}{4}$; 1, $\frac{3}{4}$, $\frac{3}{4}$, $\frac{25}{9}$, $\frac{25}{9}$.
2. $\frac{8}{17}$, $\frac{15}{17}$, $\frac{15}{8}$, $\frac{15}{17}$, $\frac{15}{8}$, $\frac{17}{15}$, $\frac{8}{17}$, $\frac{15}{17}$, $\frac{15}{8}$.
   (i) sin A, (ii) cos A, (iii) cot A, (iv) tan A.
3. $b=9$; $\frac{25}{9}$, $\frac{25}{9}$, $\frac{25}{16}$, $\frac{25}{16}$, $\frac{25}{16}$, $\frac{25}{16}$, 1, 1.
4. $\frac{AD}{AB}$, $\frac{AC}{BC}$; $\frac{BD}{BA}$, $\frac{BA}{BC}$; 8·78, 39·02.
5. $\frac{AB}{CB}$, $\frac{BD}{BA}$. 6. $\frac{BD}{DA}$, $\frac{AD}{DC}$.
7. (i) sin ABD, (ii) tan BAC, (iii) cos ACD.
8. $\frac{BC}{AC}$, $\frac{CD}{CB}$.
9. (i) tan A, (ii) cos A, (iii) From sin A. 10. $4\frac{8}{13}$.

II b. p. 15.

1. sin 37° = ·60, cos 37° = ·80, tan 37° = ·75,
   cosec 37° = 1·66, sec 37° = 1·25, cot 37° = 1·33.
2. sin 49° = ·75, cos 49° = ·66, sec 49° = 1·52, tan 49° = 1·15.
3. 58° 40′, sin 58° 40′ = ·85, tan 58° 40′ = 1·6.

4. $\sec A = 1{\cdot}94$, $\tan A = 1{\cdot}66$. 5. $\frac{49}{25}$, 1.
6. $A = 80^\circ\ 36'$, $\tan\frac{A}{2} = {\cdot}85$, $2{\cdot}15$.
7. 28°, $\cos 28^\circ = {\cdot}9$, $\sec 28^\circ = 1{\cdot}1$.
8. $\tan 40^\circ = {\cdot}84$, $\tan 20^\circ = \cot 70^\circ = {\cdot}36$. 9. $\frac{4}{3}$.
13. $BE = 8''$, $BF = 6{\cdot}9''$.

MISCELLANEOUS EXAMPLES A. p. 16.

2. $\tan 48^\circ = 1{\cdot}11$. 3. $\sin A = {\cdot}6$, $\cos C = {\cdot}6$, $A + C = 90^\circ$.
4. 120°. 5. $19^\circ\ 18'\ 18''$. 6. $63^\circ\ 30'$.
7. 10. 8. 68°, $\cdot 40$. 9. $\cdot 58$. 10. $\cdot 25$, $\cdot 26$. 11. 150°.
13. 32° nearly. 14. $8''$. 15. $2{\cdot}4$. 17. 30°, 60°, 90°.

III a. p. 20.

1. $\cdot 3256$. 2. $\cdot 5500$. 3. $\cdot 4215$. 4. $\cdot 9506$.
5. $1{\cdot}7079$. 6. $\cdot 8976$. 7. $\cdot 3025$. 8. $3{\cdot}9894$.
9. $2{\cdot}9478$. 10. $5{\cdot}9351$. 11. $4{\cdot}8642$. 12. $6{\cdot}1742$.
13. $62^\circ\ 28'$. 14. $63^\circ\ 43'$. 15. $61^\circ\ 7'$. 16. $78^\circ\ 49'$.
17. $75^\circ\ 26'$. 18. $75^\circ\ 50'$. 19. $11^\circ\ 32'$, 30°.
20. $36^\circ\ 52'$, $48^\circ\ 11'$. 21. $41^\circ\ 49'$. 22. $51^\circ\ 20'$, $71^\circ\ 34'$.

EXERCISE. p. 21.

(i) $AB = x \operatorname{cosec}\theta$, $BC = x\cot\theta$.
(ii) $AB = y\sec\phi$, $BC = y\tan\phi$.
(iii) $BC = x\tan\theta$, $AC = x\sec\theta$.
(iv) $AB = y\cos\phi$, $BC = y\sin\phi$.
(v) $AC = x\cot\theta$, $BC = x\operatorname{cosec}\theta$.
(vi) $AB = y\cos\phi$, $AC = y\sin\phi$.
(vii) $c = 17{\cdot}013$, $a = 13{\cdot}764$.
(viii) $c = 10{\cdot}946$, $a = 4{\cdot}452$.
(ix) $c = 22{\cdot}69$, $b = 10{\cdot}718$.
(x) $c = 15{\cdot}146$, $b = 11{\cdot}376$.
(xi) $b = 15{\cdot}4725$, $a = 19{\cdot}6375$.
(xii) $b = 16{\cdot}929$, $a = 11{\cdot}1132$.

III b. p. 23.

1. 3·464 in. 2. 7·66, 6·43, 11·92, 9·13 in. 3. 318·5 ft.
4. 33°. 5. 35°. 6. 4·37 ft., 8·25 ft., 41° 11′.
7. 273 ft. 8. 3·06 ft., 3·83 ft. 9. 11·28 cm. 7·71 cm.
10. 148·26 ft. 11. 17·32, 6·84, 18·79, 24·53 in.
12. 246 ft. 13. 12·86, 15·32, 19·32, 5·18 ft.

EXERCISE. p. 26.

1. 237·8 sq. in. approx. 2. 58·8 in. approx.
3. 363 sq. in. approx., 72·6(5) in. approx.

III c. p. 30.

1. 10·23 sq. in. 2. 5·23 in. 3. ·076 ft. 4. 109 ft.
5. 93·53 sq. in., 36 in. 6. 63·86 ft., 60° 34′.
7. 133·7 ft. 8. 60°, 30°, 6·93 in. 9. 14·69 sq. ft.
10. 8·86, 6·25, 7·46 cms. 11. 59° 29′. 12. 37 yds.
13. 61·9 ft. 14. 8·76 in. 15. 696 ft. 16. 42·4 mls.
17. 71·4, 79·3 ft. 18. 2630 ft. approx. 19. 1081 ft.
20. 126·6 ft. 21. 6104 sq. ft.
22. (i) 4·43 mls., (ii) 5·97 mls., (iii) 7·4 mls. 23. 39 ft.
24. 73·5 sq. ft., 30·90 ft.; 81·2 sq. in., 32·49 in.
25. 149·6 ft. 26. 140, 184 ft. 27. 1·245 mls.
28. 96·2 yds. 29. 6 miles. 30. 121 yds., E. 51° N.

MISCELLANEOUS EXAMPLES B. p. 33.

1. 7·8 cms., 6·3 cms., 8·1 cms. 2. 9·95 cms., 6·71 cms.
3. 1·40. 4. (i) 1·0724, (ii) 3·6280.
5. 41° 49′, 10·47 cms. 7. 5·14 cms., 12·86 sq. cms.
8. 84 ft. 10. $13\frac{1}{4}$ mls., N. 13° 7′ W. 11. 23° 51′, 28° 9′.
12. 26° 47′. 13. 0°, 30°. 15. 37°.
16. 1·805 in., 61° 1′, 61° 1′, 57° 58′. 17. 48° 35′, 14° 29′
18. 2° 22′. 19. 38·04 sq. in.
20. $a=5$, $\sin 2A = \cdot 71$, $\sin A = \frac{5}{13}$, $\cos A = \frac{12}{13}$.
21. 266·95 yds. 22. ·05 ins., ·0033 ins. 23. 61° 19′.
24. 31° 41′. 25. 56° 19′, 53° 8′. 26. 21·3 ft.
27. 30°, 41° 49′. 28. 3 ch. 27 links.

IV a. p. 46.

1. (1) ·9063. (2) $-$·6428. (3) -1·0038.
(4) $-$·7813. (5) ·6691. (6) $-$·7536.
(7) ·8129. (8) $-$·1432. (9) $-$·5878.
(10) 1·9841. (11) 3·6280. (12) 4·4919.
2. (1) $115° 1'$, $295° 1'$. (2) $19° 7'$, $199° 7'$.
(3) $63° 5'$, $116° 55'$. (4) $112° 46'$, $247° 14'$.
(5) $241° 1'$, $298° 59'$. (6) $18° 43'$, $198° 43'$.
3. $35°$, $215°$.
7. (1) $34° 31'$ or $145° 29'$. (2) $51° 19'$. (3) $113° 35'$.
9. (1) $30°$, $150°$. (2) $53° 8'$, $126° 52'$, $210°$, $330°$.
(3) $168° 41'$, $348° 41'$, $68° 12'$, $248° 12'$.
10. (1) $18°$. (2) $10°$.
11. (1) $36°$. (2) $60°$. (3) $36°$ or $60°$.

EXERCISE. p. 49.

(i) $1, \infty, 0$. (ii) $0, -1, 0$. (iii) $\infty, -1, -\infty$.
(iv) $-1, 0, \infty$. (v) $-1, -\infty, 0$.

IV b. p. 54.

2. (1) $45°$, $225°$. (2) $135°$, $315°$.
5. (1) $\theta=90°$ or $270°$. $OP=4$. (2) $\theta=0°$ or $180°$, $OP=5$.
6. $45°$, 312·5 ft., $9° 20'$, $80° 40'$.

V a. p. 59.

1. $75° 31'$. 2. 4·23. 3. $112° 53'$. 4. 6·47 m., 4·02 m.
6. $A=41° 24'$, $B=55° 47'$, $C=82° 49'$. 7. 4·86 ft., 1·55 ft.
8. $a=7·41$, $B=80° 49'$, $C=52° 11'$. 9. $81°$, $19°$.
10. 8·83 ft. 11. 6·14. 12. $110° 29'$.
14. $B=27° 50'$, $C=37° 10'$.
15. $4x^2-32x+31=0$, 6·87, 1·13 miles.

MISCELLANEOUS EXAMPLES C. p. 61.

1. (1) 75° 58′. (2) 1·134″. **2.** 69° 18′, 110° 42′.
4. 225·3 ft. **7.** 9·06 sq. in., 3·71, 5·54 in. **8.** 185·8 yds.
10. 41·5 ft. **11.** $\cos\theta = \dfrac{1}{\pm\sqrt{\tan^2\theta+1}}$. **12.** 1026 ft.
13. 18° 56′, 8·14 in.
16. ·018, ·019, ·021, ·026, ·035, ·054, ·102, ·292. Increases from 5·67 to ∞.
18. 36° 52′, 146° 19′, 216° 52′, 326° 19′. **20.** 109° 6′.

VI a. p. 71.

1. $\frac{4}{5}$, $\frac{3}{5}$, 1, 0, $\frac{7}{25}$, $\frac{24}{25}$. **2.** ·9428, ·9683, ·5585.
3. ·9484. **5.** (i) ·5150. (ii) $-$·1908. **7.** $\dfrac{\sqrt{6}-\sqrt{2}}{4}$.
12. (i) ·5878. (ii) ·8090. **14.** $(\cos A+\sin A)(\cos B-\sin B)$.
15. $\cos A\cos B\cos C-\cos A\sin B\sin C-\cos B\sin A\sin C$
$-\cos C\sin A\sin B$.

VI b. p. 73.

2. ·5095. **3.** $2-\sqrt{3}$. **7.** $\dfrac{\cot A\cot B+1}{\cot B-\cot A}$. **8.** $\frac{1}{3}$.
10. 1. **14.** 120 ft.

VI c. p. 76.

1. $\frac{3}{5}$, $\frac{24}{25}$, $-\frac{7}{25}$. **2.** $\pm$·7333, $-$·6800, $\pm$1·078.
3. ·7660, ·6428. **5.** $\frac{4}{3}$. **8.** $\pm\frac{4}{5}$. **9.** $\pm\frac{3}{4}$.
11. ·4695, ·8829. **12.** ·3640. **13.** $\pm\frac{1}{2}$. **14.** 2.
16. (1) 30°, 150°, 210°, 330°.
(2) 0°, 30°, 150°, 180°, 210°, 330°.
17. $8\cos^4 a-8\cos^2 a+1$. **18.** a.
20. $\frac{1}{2}(\cos 2a+\cos 2\beta)$, $-$·3288. **21.** $\dfrac{a+b}{a-b}$.
23. Projection equals $r+r\cos\theta$.
24. Height equals $r-r\cos\theta$.

VI d. p. 79.

1. $36° 52'$. 2. $103° 17'$. 3. $114° 18'$. 4. $16° 16'$.
5. $\sqrt{p^2+q^2}$.

VI e. p. 81.

1. $\sin 4\theta + \sin 2\theta$. 2. $\cos 4\theta + \cos 2\theta$. 3. $\frac{1}{2}(\cos 2\theta - \cos 4\theta)$.
4. $\sin 4\theta - \sin 2\theta$. 5. $\frac{1}{2}(\sin 3A - \sin A)$.
6. $\frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$.
7. $\frac{1}{2}\{\cos 2(A+B) + \cos 2(A-B)\}$. 8. $\frac{1}{2}(\cos 4\theta - \cos 6\theta)$.
9. $1 - \sin 50°$. 10. $\cos 70° + \cos 10°$.
11. $\cos 10° - \cos 30°$. 12. $\frac{1}{2}\{\cos 80° + \cos 20°\}$.
13. $\sin 2A + \sin 2B$. 14. $\cos 3(A+B) + \cos(A-B)$.
15. $\sin A$. 16. $\frac{1}{2}(\cos 2a - \cos 4a)$.

VI f. p. 82.

1. $2 \sin 2A \cos A$. 2. $2 \cos 2A \sin A$. 3. $2 \cos 2A \cos A$.
4. $2 \sin 2A \sin A$. 5. $2 \cos \frac{3\theta}{2} \sin \frac{\theta}{2}$.
6. $-2 \sin \frac{5\theta}{2} \sin \frac{\theta}{2}$. 7. $2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$.
8. $2 \cos(\alpha+\beta) \cos(\alpha-\beta)$. 9. $2 \sin(\alpha+\beta) \sin(\beta-\alpha)$.
10. $2 \sin 18° 30' \cos 4° 30'$. 11. $2 \sin 36° 30' \sin 4° 30'$.
12. $\sin 41° + \sin 78° = 2 \sin 59° 30' \cos 18° 30'$.
13. $2 \cos 30° 30' \cos 12° 30'$.
21. $\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$ and in a triangle

$$C = 180 - (A+B), \quad \therefore \sin C = \sin(A+B).$$

MISCELLANEOUS EXAMPLES D. p. 84.

1. $m = m'$. 2. $\frac{3}{4}$, $-\frac{1}{2}$, $\frac{1}{2}$. 3. 3·90 ft. 3·52 ft.
4. 16·16 ft. 5. Square and add. 6. $\frac{63}{65}$, $75° 45'$.
7. $c \cos A \pm \sqrt{a^2 - c^2 \sin^2 A}$, 16·25 cms. 9. 5·32 ft.
10. $\tan\theta_1 \tan\theta_2 = -1$. See Qu. 1.
11. $x = -\frac{\sin\alpha}{\cos(\alpha+\beta)}$, $y = \frac{\cos\alpha}{\cos(\alpha+\beta)}$. 15. $\frac{27}{179}$, $8° 34'$

VII a. p. 90.

1. 1, 2, −1, −2, 3; 0, −4, 4, −3, −1.
2. ·6045, 2·6045, $\bar{1}$·6045, $\bar{3}$·6045, 4·6045.
3. 21·74, ·02174, 2·174, 21740, ·002174, 217·4, ·2174.
4. ·6020, ·6990, ·7781, ·9030, ·9542, 1·0791, 1·1761, 1·2040, 1·2552, 1·3010.
   ·845, 1·0395, 1·146, 1·2781. 1·113. 1·226.

VII b. p. 91.

1. 2·6749, ·6754, $\bar{1}$·4570, 5·6590, 1·9428, $\bar{3}$·5710.
2. 2·969, 5569, ·7314, 16500, ·004839.

VII c. p. 94.

1. 1·059. **2.** 10·89. **3.** 7·750. **4.** 173·2.
5. ·2086. **6.** ·04223. **7.** 127·8. **8.** ·05551.
9. 12·95. **10.** 13·38. **11.** ·8950. **12.** ·3840.
13. ·8555. **14.** 4·108. **15.** ·00006101. **16.** ·2601.
17. 1·975. **18.** ·005610. **19.** ·3163. **20.** 14.
21. 6. **22.** 39·98. **23.** ·95. **24.** £425. 15*s*.
25. 18. **26.** 22·99. **27.** 2214 sq. ft., 9790 cu. ft.
28. 121·5. **29.** 304·2, ·01991. **30.** ·028, ·00782.
31. (1) 4, (2) −4. **32.** 3·484. **33.** 7757×10^{13}.
34. ·938. **35.** 2·442, −·511. **36.** 2·254. **37.** ·09281.
38. 305·5. **39.** 33130. **40.** 360·2. **41.** ·00005903.
42. 8028×10^{8}. **43.** ·01848. **44.** ·2384.
45. ·0000003243. **46.** 9888×10^{8}.

VII d. p. 97.

1. $\bar{1}$·9219, $\bar{1}$·7112, ·2614, $\bar{1}$·8611, ·4453, $\bar{1}$·9224.
2. (1) 17° 25′, (2) 65° 2′, (3) 75° 24′, (4) 82° 22′, (5) 21°.
3. (1) ·6029, (2) −·3822, (3) ·4276.
4. 37° 11′, 142° 49′. **5.** 16° 28′. **6.** 241.
7. −·2831. **8.** 150400 sq. ft. **9.** 22° 16′.
10. 81° 12′. **11.** ·2004. **12.** ·0393.
13. 1·518. **14.** $7·958 \times 10^{-6}$. **15.** ·1803.
16. 12·03. **17.** 15·68 grams wt. **18.** 9·475 cms.
19. ·01289. **20.** 83° 53′. **21.** 5·780.

MISCELLANEOUS EXAMPLES E. p. 100.

1. $27° 45'$. 2. $\pm\cdot 7018$.

3. (i) $10^{\cdot 5} = \sqrt{10} = 3$ approx., (ii) $10^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{10}} = \cdot 56$ approx., (iii) $(\cdot 35)^2 = \cdot 12$ approx.

4. $14° 2'$, $45°$, $194° 2'$, $225°$. 5. 3·16 sq. cms.

6. $\log \cos\theta = \log \sin(90° - \theta)$; $\log \tan\theta = \log \sin\theta - \log \cos\theta$.

7. 642·2. 8. $78° 28'$. 10. 29·4 in., 59·4 sq. in.

11. (i) 27·01 sq. ft., (ii) 5·106 ft.

12. 65·1 ft. 14. $x = \tan 6° = \cdot 1$.

16. 4·193 in.; $XY = 5\{\cos a + \cos(90° - a)\} = 10 \cos 45° . \cos(45° - a)$, $\therefore$ XY least when $a = 0$, greatest when $a = 45°$.

18. $\cdot 3\ {}^{\circ}/_{\circ}$. 19. 20·7.

VIII a. p. 108.

1.	$A = 29° 56'$,	$B = 42° 3'$,	$C = 108° 1'$.
2.	$C = 77° 31'$,	$a = 51\cdot 4$,	$b = 77\cdot 2$.
3.	$A = 61° 21'$,	$a = 25\cdot 2$,	$c = 19\cdot 7$.
4.	$A = 33° 26'$,	$B = 65° 10'$,	$c = 474$.
5.	$A = 111° 24'$,	$B = 22° 6'$,	$a = 36\cdot 55$.
6.	$B = 99° 13'$,	$C = 44° 23'$,	$b = 46\cdot 6$.
or	$B = 7° 59'$,	$C = 135° 37'$,	$b = 6\cdot 56$.
7.	$B = 40° 52'$,	$C = 32° 8'$,	$c = 254$.
8.	$A = 78° 48'$,	$B = 53° 10'$,	$C = 48° 2'$.
9.	$C = 35° 38'$,	$a = 5\cdot 80$,	$b = 3\cdot 93$.
10.	$A = 26° 22'$,	$C = 31° 38'$,	$b = 83\cdot 18$.
11.	$C = 39° 11'$,	$a = 2663$,	$c = 2001$.
12.	$A = 44° 49'$,	$B = 60°$.	
13.	$B = 54° 56'$,	$C = 83° 4'$,	$c = 209$.
or	$B = 125° 4'$,	$C = 12° 56'$,	$c = 47\cdot 2$.

14. $89° 55'$ or $15° 21'$. 15. 567 yds. 16. 3·9 ft.

17. $120°$. 18. $A = 60°$, $b = 3\cdot 84$ in., $c = 4\cdot 76$ in.

19. $A = 95° 12'$, $C = 64° 13'$. 20. N. 30° E.

21. (i) 1·27 miles, (ii) 5·51 miles.

22. 10·1 ft. 23. 26·1 yds. 24. 978 yds., 2546 yds.

25. 31·3 miles N., 9·2 miles W., AB = 32·6 miles N. 16° 18′ W.

26. 12.46 p.m. 27. 3073 yds.

28. 263 yds. 29. 13 sea miles.

30. AP = 1639 m., BP = 1594 m.; ABP obtuse, AP = 1639 m., BP = 1058 m. **31.** 264·2 yds., 235·8 yds.
32. 8·76 miles. **33.** 1085 yds.
34. 14·88 miles N. 1° 49′ W. **35.** $40\sqrt{3} = 69 \cdot 3$ yds.
36. 5·7 miles, 5·57 miles.

VIII b. p. 114.

2. 1·61. **3.** 318. **4.** 81 ft. **5.** 30.

VIII c. p. 118.

1. 2·7 sq. in. **2.** 4·403. **3.** 24·7. **4.** 10·7.
5. 79·8 ft., 20,000 sq. ft. **6.** 448, 122 links.
7. 12 : 5, 5 : 18, 3 : 2. **11.** $\dfrac{a^2 \sin B \sin C}{2 \sin A}$.

MISCELLANEOUS EXAMPLES F. p. 119.

1. 1779 yds., 992·6 yds. **3.** $7 \cdot 228 \times 10^{10}$.
4. 70° 32′. **6.** 76·9 chains; S. 72° 38′ E.
7. 3·1 miles. **8.** Project on the horizontal side.
9. 84° 25′. **10.** 4·243 cms.
11. 65·28, ·06 yds. **12.** 8·7 yrs.
14. A = 53° 8′, B = 106° 16′, C = 20° 36′, c = 10·99.
15. 2, 43, 42. **16.** 1317 yds.
17. A = 56° 5′, C = 92° 10′; or A = 123° 55′, C = 20° 20′.
18. 14·12″. **20.** 203·6 c. in. **21.** 8·47″.
22. $2 \cos 45° \cos (45° - \theta)$. Max. when $\theta = 45°$. Min. when $\theta = 0°$.
23. 4°/∘. **24.** 50° 12′. **25.** 5·29″, 39° 6′.

IX a. p. 125.

1. $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{36}, \dfrac{179\pi}{432}$.
2. 45°, 120°, 128° 34′ $17\frac{1}{7}$″, 300°. **3.** 1·26, ·83, 2·34.
4. 71° 3′, 36° 6′. **5.** 8·4 cms. **6.** ·05.
7. $\dfrac{2\pi}{3}$. **8.** $8\frac{8}{9}$ in. **9.** 45° 50′. **10.** 135°, $\dfrac{3\pi}{4}$.
11. 3960 miles approx. **12.** ·0398 secs. **13.** 3·4 cms.
14. 2° nearly. **15.** 38·5. **17.** 2600 miles.
18. 2·08 in., 1·88 in., 1·26 in. **19.** 39 ft. 9 in.

IX b. p. 133.

1. 314·16 sq. in. 2. 11·65 cms.
3. 8·38 sq. ft., 1·45 sq. ft. 4. 2165 miles.
5. 860000 miles. 6. 16·7′. 7. 21·3 miles.
8. 34·8 miles. 11. ·00582. 14. 57·3 in.
15. 31·416 in., 38·9 sq. in.

MISCELLANEOUS EXAMPLES G. p. 134.

1. 3560, 2517 miles. 2. 54 ins. 3. 29 ft.
5. 9·27, 15·86 ins. 6. 0, 3 cm., 3 cm., 1, 1, ·0874 radians, ·0872 = sin 5°, ·0875 = tan 5°. 7. 645 miles per hr.
8. 2504 ft. 9. 3·73 cms. 10. 11 miles.
11. $AP' = 2r \sin \frac{x}{2}$. Describe a circle whose radius is the distance from A to graduation 60°. An angle of $x°$ is subtended at the centre of the circle by a chord whose length is the distance from A to the graduation x.
12. 26·7 yds. 13. 90° 29′, 84·5 ft. 14. ·1587.
15. 30·5 miles. 16. ·007272.
17. (i) 2092, (ii) 1162, (iii) 1168.
19. 78·4 chains. 20. 75·75 yds.
24. $\sin(60-\theta) = \frac{3\sqrt{3}-2}{8}$, $\theta = 36° \, 24'$.
25. $R = 1{\cdot}23$, $h = 3{\cdot}8$ ft. 26. $b \cos\theta - a \sin\theta$.
27. $x = a\cos\theta$, $y = a\sin\theta$. 28. $r \cos\frac{(\beta \sim a)}{2} \sec\frac{a+\beta}{2}$.
29. $\tan\frac{C}{2} = \frac{4}{7}$. 30. 69°.

X a. p. 146.

1. 35° 16′. 2. 60°. 3. 7·810 in., 39° 48′.
4. 27° 19′, 20° 8′. 5. 33° 4′.
6. (1) 11·47 in. (2) 2·97 in. (3) 8° 32′.
7. 45°, $\frac{a\sqrt{2}}{2}$. 8. 23° 56′. 9. 5·57 ft., 29° 30′.
10. 7·27 in., 37° 18′. 11. 54° 44′, 2·89 in. 12. 12° 20′.
13. 5475 ft., 12° 40′. 14. 31°, 64° 37′, 73° 24′.
16. 159·15 sq. ft. 17. $\sqrt{38}$, $\sqrt{29}$, $\sqrt{13}$ ft.

X b. p. 151.

1. 164 yds. 2. 800 yds. 3. 172 yds.
4. 106 yds. 5. 280 ft. 6. 2·24 miles.
7. 6510 ft. 8. 56° approx. 9. 2690 yds. nearly.
10. 2874 ft. nearly. 11. 5 miles per hr. 12. 1·412 : 1.
13. 2092 ft. 14. 51° nearly. 15. 2193 ft.
16. 8·63 miles, N. 17° 54′ E. 17. 17169 cu. yds.
18. 14·9 sq. in. 19. 37° 30′. 20. 1014 sq. ft.

TRIANGULATION. p. 156.

DE = 366 ft., FE = 412 ft., FG = 274 ft., EG = 308 ft.

EXERCISE. p. 156.

PQ = 112 ft., QR = 148 ft., RP = 102 ft.

MISCELLANEOUS EXAMPLES H. p. 157.

1. (1) 105·5, (2) 849·4.
2. (1) 21·5 chains E., (2) 9·2 chains N., (3) 23·4 chains, (4) N 66° 45′ E.
3. 77° 10′, 139° 21′, 84° 16′, 59° 13′, 37·45 sq. ft. 4. 18·9 ft.
6. (1) 8·116, (2) ·4005. 7. 6·83 miles per hr.
8. DA = 50 ft., AE = 160 ft., EB = 194 ft., BD = 120 ft.; 112° 23′, 67° 37′, 128° 39′, 51° 21′, 14440 sq. ft.
10. 13° 54′ with a line going E. and W. 11. 23660 tons.
12. 3600 cu. in. 13. 12, 16, 20 ft. 14. 17° 48′ nearly.
17. 376 ft. 20. 28,000 miles. 21. 17 ft.
25. $-4\sin^2\frac{A}{2}$. 28. 120°.

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